

DERIVED CATEGORIES AS ALGEBRAIZATIONS

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Abstract. Using classical ideas from universal algebra, we describe a common framework for constructing and characterizing various types of derived ∞ -categories. Examples include the derived ∞ -categories of certain abelian categories, the ∞ -category of derived commutative rings, and certain ∞ -categories of synthetic spectra.

Contents

0. Introduction	2
0.1. Overview.	2
0.2. Acknowledgements.	4
0.3. Conventions.	4
1. Categorical preliminaries	6
1.1. Partial adjoints.	6
1.2. Density.	9
2. Algebraic structure.	13
2.1. $\mathcal{X}$ -objects.	14
2.2. Theorization	16
2.3. Algebraization	23
3. Functoriality of theorization	28
3.1. Variance	28
3.2. Contravariance	31
3.3. Covariance.	37
3.4. Derived functors	42
4. Spectral algebra.	44
4.1. Theories	44
4.2. Theorization	49
4.3. Stability	51
4.4. Connectivity	52
5. Banimation	56
5.1. Animation.	57
5.2. Banimation.	58
5.3. Abelian categories I	65
5.4. Commutative rings	65
5.5. Abelian categories II	70
6. Even synthesis.	72
6.1. Filtered spectra.	72
6.2. The perfect even filtration	74
6.3. The even filtration.	77
6.4. Conclusion.	82
A. Geometric stacks	85
A.1. Stacks	85
A.2. Geometricity	86
A.3. Unipotence	94

Ch. 0. Introduction

§0.1. Overview. Sometimes, when we’re working with a category $\mathcal{C}$ of an algebraic nature, we end up interested in a larger “derived ∞ -category” $\mathcal{D}$ into which $\mathcal{C}$ embeds.

The prototypical example for us is where $\mathcal{C}$ is a Grothendieck abelian category, like the category $\mathbf{Ab}$ of abelian groups, and $\mathcal{D}$ is the unbounded derived ∞ -category $D(\mathcal{C})$ of Lurie. This construction arose via, and is now core to, the study of homology and cohomology, in their many incarnations. We refer to the overview of the earlier history of homological algebra by Weibel [40] and the summary of the ∞ -categorical perspective in the work of Gaitsgory–Lurie [15] for more information.

There are other examples too, including ones where $\mathcal{C}$ is not additive, though they do not come from the same formalism. For instance, $\mathcal{C}$ might be the category $\mathbf{CRing}$ of commutative rings and $\mathcal{D}$ the ∞ -category $\mathbf{DAlg}$ of derived commutative rings, defined by Mathew. This is closely tied to the subject of derived algebraic geometry. Connective objects of $\mathbf{DAlg}$ are equivalent to animated commutative rings, which are modelled by simplicial commutative rings and correspond to affine derived schemes [14, 29]. Coconnective objects of $\mathbf{DAlg}$, on the other hand, correspond to affine stacks in the sense of Töen and are modelled by cosimplicial commutative rings [30, 38]. This structure is applied for instance in Petrov’s work on Hodge decompositions in positive characteristic [31], as well as in several other studies of algebraic de Rham cohomology and its relatives, e.g. [2, 3, 8, 11, 19, 35].

There are even examples where $\mathcal{C}$ is itself actually a non-classical ∞ -category, such as the ∞ -category $\mathbf{Spt}$ of spectra, for which $\mathcal{D}$ may be taken to be any of the various ∞ -categories of synthetic spectra introduced by Pstrągowski [34]. The version of this construction based on complex bordism (or equivalently on evenness [33]), which was also formulated from a different perspective by Gheorghe–Krause–Isaksen–Ricka [16], is of particular interest due to its connection to the theory of motives; e.g., it gives a purely homotopy-theoretic model for a part of Morel–Voevodsky’s motivic stable homotopy theory over the field of complex numbers. More directly, it can be regarded as a categorical enhancement of the Adams–Novikov spectral sequence, articulating some of the essence of chromatic homotopy theory in a precise and structured way and having concrete applications to calculations in stable homotopy theory [13, 21].

This paper describes a common framework for constructing and characterizing a range of the types of derived ∞ -categories enumerated above, including both non-additive and non-classical cases of $\mathcal{C}$. Conceptually, this articulates a precise connection among their prior, disparate definitions. Practically, it provides a different way of working with them and a method for generalizing them.

Our approach, more specifically, will be to identify these constructions as instances of an extremely general device from universal algebra. A rough sketch of it is that, in each case:

1. There is a more basic ∞ -category $\mathcal{X}$ of “derived” objects, such that objects of $\mathcal{D}$ can be viewed as objects of $\mathcal{X}$ equipped with some kind of algebraic structure: that is, there is a canonical functor $u_{\mathcal{D}} : \mathcal{D} \rightarrow \mathcal{X}$, belonging to a particular class $\mathfrak{a}$ of functors to $\mathcal{X}$ which codifies the notion of a forgetful functor for algebraic structure.
2. Let $i : \mathcal{C} \rightarrow \mathcal{D}$ denote the embedding and let $u_{\mathcal{C}} := u_{\mathcal{D}} i : \mathcal{C} \rightarrow \mathcal{X}$. Then i exhibits $u_{\mathcal{D}}$ as the universal approximation to $u_{\mathcal{C}}$ by a member of $\mathfrak{a}$. In this sense, $\mathcal{D}$ is an “algebraization” of $\mathcal{C}$ relative to $\mathcal{X}$. In some cases, this holds instead when $\mathcal{C}$ is replaced by a well chosen subcategory $\mathcal{C}' \subseteq \mathcal{C}$.
3. There is a more basic description of the functor $u_{\mathcal{C}}$, which does not require knowledge of $\mathcal{D}$. Thus, 2 provides an a priori characterization of $\mathcal{D}$ (together with i and $u_{\mathcal{D}}$). Furthermore, this can be used to give an explicit construction of $\mathcal{D}$.

The above sketch is substantiated in the body of the paper as follows. In Ch. 2, we’ll discuss two ways to axiomatize algebraic structure relative to an arbitrary presentable ∞ -category $\mathcal{X}$ (Definition 0.1.3 below is one example) and the problem of constructing universal approximations to ∞ -categories over $\mathcal{X}$ by algebraic structure in these senses; as alluded to above, these are classical topics in universal algebra, going back in particular to work of Lawvere and Linton. In Ch. 4, we’ll make these general ideas more explicit in the special case that $\mathcal{X}$ is a compactly generated stable ∞ -category (e.g. $\mathbf{Spt}$), which is the setting of interest for the examples of derived ∞ -categories studied in the remainder of the paper. In Ch. 5,

we'll formulate a construction that we call *bianimation* and show that it reproduces, for example, $D(\mathbf{Ab})$ and $\mathbf{DAlg}$, in the manner outlined above (in particular proving Theorems 0.1.1 and 0.1.4 below). Finally, in Ch. 6, we'll discuss such a perspective on the version of synthetic spectra studied in [33] (in particular proving Theorems 0.1.2 and 0.1.5 below).

Supplementing the main narrative just described are: Ch. 1, which reviews a couple of categorical concepts, including density, which permeates the paper; Ch. 3, which is a kind of appendix to Ch. 2, primarily concerning how algebraic structure over different presentable ∞ -categories may be related; and Ch. A, an actual appendix to the paper, which reviews some notions from algebraic geometry that are relevant to our discussion of derived commutative rings.

To make this overview more concrete, let us state the following characterizations of the ∞ -category $\mathbf{DAlg}$ of derived commutative rings and the ∞ -category $\mathbf{Syn}^{\mathrm{ev}}$ of even MU-based synthetic spectra that will be established here. We continue to denote the ∞ -category of spectra by $\mathbf{Spt}$; we let $\mathbf{Spt}^{\mathrm{ev}} \subseteq \mathbf{Spt}$ denote the full subcategory of spectra with homotopy concentrated in even degrees; we denote the ∞ -category of filtered spectra by $\mathbf{Fil}$; and we denote the ∞ -category of spaces by $\mathbf{Spc}$.

Theorem 0.1.1. *For $n \in \mathbb{Z}_{\geq 0}$, let $B^n \mathbb{G}_a : \mathbf{CRing} \rightarrow \mathbf{Spc}$ be the functor sending a commutative ring R to the Eilenberg–MacLane space $B^n R = K(R, n)$ associated to the underlying additive group of R . Note that we have canonical pullback diagrams*

$$\begin{array}{ccc} B^n \mathbb{G}_a & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & B^{n+1} \mathbb{G}_a, \end{array}$$

where 1 denotes the constant functor with value the one point space. Let $\mathcal{T} \subseteq \mathbf{Fun}(\mathbf{CRing}, \mathbf{Spc})$ be the full subcategory generated under finite products by the functors $B^n \mathbb{G}_a$, and let $\mathbf{Fun}'(\mathcal{T}, \mathbf{Spc}) \subseteq \mathbf{Fun}(\mathcal{T}, \mathbf{Spc})$ be the full subcategory of functors that preserve finite products and the pullbacks displayed above. Then there is a canonical equivalence of ∞ -categories $\mathbf{DAlg} \simeq \mathbf{Fun}'(\mathcal{T}, \mathbf{Spc})$.

Theorem 0.1.2. *Consider the functors $B^n \Omega^{\infty+2m} : \mathbf{Spt}^{\mathrm{ev}} \rightarrow \mathbf{Spc}$ for $n \in \mathbb{Z}_{\geq 0}$ and $m \in \mathbb{Z}$, together with the canonical pullback diagrams*

$$\begin{array}{ccc} B^n \Omega^{\infty+2m} & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & B^{n+1} \Omega^{\infty+2m}, \end{array}$$

where 1 denotes the constant functor with value the one point space. Let $\mathcal{T} \subseteq \mathbf{Fun}(\mathbf{Spt}^{\mathrm{ev}}, \mathbf{Spc})$ be the full subcategory generated under finite products by the functors $B^n \Omega^{\infty+2m}$, and let $\mathbf{Fun}'(\mathcal{T}, \mathbf{Spc}) \subseteq \mathbf{Fun}(\mathcal{T}, \mathbf{Spc})$ be the full subcategory of functors that preserve finite products and the pullbacks displayed above. Then there is a canonical equivalence of ∞ -categories $\mathbf{Syn}^{\mathrm{ev}} \simeq \mathbf{Fun}'(\mathcal{T}, \mathbf{Spc})$.

Definition 0.1.3. Let $\mathcal{X}$ be a presentable ∞ -category. We say that a functor $u : \mathcal{Y} \rightarrow \mathcal{X}$ is *algebraic* if it is conservative, admits a left adjoint, and creates sifted colimits (or equivalently if $\mathcal{Y}$ is presentable and u is monadic and preserves sifted colimits).

Theorem 0.1.4. *Let $u_{\mathbf{DAlg}} : \mathbf{DAlg} \rightarrow \mathbf{Spt}$ be the forgetful functor, and let $u_{\mathbf{CRing}} : \mathbf{CRing} \rightarrow \mathbf{Spt}$ be the restriction thereof, carrying a commutative ring R to the Eilenberg–MacLane spectrum associated to underlying additive group of R . Let $u : \mathcal{Y} \rightarrow \mathbf{Spt}$ be an algebraic functor and let $v : \mathbf{CRing} \rightarrow \mathcal{Y}$ be a functor over $\mathbf{Spt}$. Then v extends uniquely to a functor $v' : \mathbf{DAlg} \rightarrow \mathcal{Y}$ over $\mathbf{Spt}$.*

Theorem 0.1.5. *Recall that we have a forgetful functor $u_{\mathbf{Syn}^{\mathrm{ev}}} : \mathbf{Syn}^{\mathrm{ev}} \rightarrow \mathbf{Fil}$, and that this restricts to the Postnikov filtration functor $u_{\mathbf{Spt}^{\mathrm{ev}}} = \tau_{\geq 2*} : \mathbf{Spt}^{\mathrm{ev}} \rightarrow \mathbf{Fil}$. Let $u : \mathcal{Y} \rightarrow \mathbf{Fil}$ be an algebraic functor and let $v : \mathbf{Spt}^{\mathrm{ev}} \rightarrow \mathcal{Y}$ be a functor over $\mathbf{Fil}$. Then v extends uniquely to a functor $v' : \mathbf{Syn}^{\mathrm{ev}} \rightarrow \mathcal{Y}$ over $\mathbf{Fil}$.*

The forgetful functors $u_{\mathbf{DAlg}} : \mathbf{DAlg} \rightarrow \mathbf{Spt}$ and $u_{\mathbf{Syn}^{\mathrm{ev}}} : \mathbf{Syn}^{\mathrm{ev}} \rightarrow \mathbf{Fil}$ are themselves algebraic (this is a

more substantial fact in the former case; our discussion will include a review of the original explanation of it as well as another one). Thus, Theorems 0.1.4 and 0.1.5 are universal characterizations of the form sketched above, taking $\mathfrak{a}$ to be the class of algebraic functors. Though it's less apparent, we'll see that Theorems 0.1.1 and 0.1.2 may also be understood as such characterizations, for a different, but related, choice of $\mathfrak{a}$. The relation will allow us to deduce Theorems 0.1.4 and 0.1.5 from Theorems 0.1.1 and 0.1.2.

As alluded to above, in addition to covering existing examples of derived ∞ -categories, the ideas in this paper may be used to construct new variants. Investigations of this kind will appear in future work. For one, forthcoming joint work with Devalapurkar–Hahn–Senger will introduce a notion of *synthetic* $\mathbb{E}_\infty$ -ring, comprising a derived ∞ -category for $\mathbb{E}_\infty$ -rings that is closely related to the even filtration of [17]; we note that this example is simultaneously non-additive and non-classical in nature.

The framework of Ch. 4 also seems apt for studying Koszul duality between spectral algebraic structures, and this should give another mechanism for new constructions. We plan to take this up too in a sequel.

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§0.3. Conventions.

1. We fix three inaccessible cardinals $\kappa < \kappa' < \kappa''$, and we use the words *small* and *large* to refer to κ -small and κ' -small objects in the standard manner. Generically, objects other than ∞ -categories are assumed to be small, while ∞ -categories are assumed only to be large, with the caveat that diagrams in ∞ -categories (in particular those indexing limits and colimits) fall under the first rule.
2. We write $\mathbf{Spc}$ for the ∞ -category of spaces and $\mathbf{SPC}$ for the ∞ -category of large spaces.
3. We write $\mathbf{Set}$ for the category of sets. We freely regard ordinary categories as ∞ -categories and in particular sometimes identify $\mathbf{Set}$ with the full subcategory of $\mathbf{Spc}$ spanned by the 0-truncated spaces.
4. We write Σ for the full subcategory of $\mathbf{Set}$ spanned by the finite sets.
5. We write $\mathbf{Spt}$ for the ∞ -category of spectra.
6. We write $\mathbf{Cat}$ for the ∞ -category of large ∞ -categories, $\mathbf{Cat}^{\text{sm}}$ for the full subcategory of $\mathbf{Cat}$ spanned by the essentially small ∞ -categories, and $\mathbf{Pr}^{\text{L}}$ (resp. $\mathbf{Pr}^{\text{R}}$) for the subcategory of $\mathbf{Cat}$ spanned by the presentable ∞ -categories and the left (resp. right) adjoint functors.
7. We write Δ for the simplex category, i.e. the category of finite, nonempty linearly ordered sets and order preserving maps between them, and Δ_s for the semisimplex category, i.e. the wide subcategory of Δ comprising the injective maps. For n a nonnegative integer, we let $[n] \in \Delta$ be $\{0, \dots, n\}$ with its standard linear ordering. *Geometric realizations* refer to colimits of simplicial diagrams, i.e. diagrams indexed by Δ^{op} , and *totalizations* refer to limits of cosimplicial diagrams, i.e. diagrams indexed by Δ . We recall that the inclusion functor $\Delta_s \rightarrow \Delta$ is coinitial.
8. Let $f : \mathcal{C} \rightarrow \mathcal{D}$ be a functor of ∞ -categories.
 - a. Depending on the context, we may write either $f^{\text{op}} : \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}^{\text{op}}$ or just $f : \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}^{\text{op}}$ to denote the induced functor on opposites.
 - b. We may write f^* to denote restriction of functors along f , and we may write $f_!$ to denote left Kan extension of functors along f .
9. Let $\mathcal{C}$ be an ∞ -category and let $C \in \mathcal{C}$. We write $\mathcal{C}_{/C}$ (resp. $\mathcal{C}_{\backslash C}$) for the slice ∞ -category whose objects are maps $\phi : C' \rightarrow C$ (resp. $\phi : C \rightarrow C'$) in $\mathcal{C}$. In some situations, we abbreviate and refer to such a map just by its domain (resp. codomain) C' , e.g. writing $C' \in \mathcal{C}_{/C}$ (resp. $C' \in \mathcal{C}_{\backslash C}$); it should be clear from our notation when we do this. When we do this and then subsequently need to make explicit the

map to (resp. from) C , we refer to it as the *structure map* of C' .

10. Let $\mathcal{C}$ be an ∞ -category.

- a. We write $j_{\mathcal{C}} : \mathcal{C} \rightarrow \text{Fun}(\mathcal{C}^{\text{op}}, \text{Spc})$ for the Yoneda embedding, carrying $C \in \mathcal{C}$ to its represented functor $\text{Map}_{\mathcal{C}}(-, C)$, and we write $j_{\mathcal{C}}^{\vee} : \mathcal{C}^{\text{op}} \rightarrow \text{Fun}(\mathcal{C}, \text{Spc})$ for the dual construction, carrying $C \in \mathcal{C}$ to its corepresented functor $\text{Map}_{\mathcal{C}}(C, -)$. In the case that $\mathcal{C}$ is locally small, we may replace Spc by Spc here.
- b. We write $\mathcal{P}(\mathcal{C})$ for the full subcategory of $\text{Fun}(\mathcal{C}^{\text{op}}, \text{Spc})$ generated under colimits by the representable functors, and refer to objects of $\mathcal{P}(\mathcal{C})$ as *presheaves* on $\mathcal{C}$. Dually, we write $\mathcal{P}^{\vee}(\mathcal{C})$ for the full subcategory of $\text{Fun}(\mathcal{C}, \text{Spc})$ generated under colimits by the corepresentable functors, and refer to objects of $\mathcal{P}^{\vee}(\mathcal{C})$ as *copresheaves* on $\mathcal{C}$. If $\mathcal{C}$ is locally small, we may replace Spc with Spc . If $\mathcal{C}$ is essentially small, then every functor from $\mathcal{C}^{\text{op}}$ (resp. $\mathcal{C}$) to Spc is a presheaf (resp. copresheaf). If $\mathcal{C}$ is accessible (resp. $\mathcal{C}^{\text{op}}$ is accessible) then a functor from $\mathcal{C}$ (resp. $\mathcal{C}^{\text{op}}$) to Spc is a copresheaf (resp. presheaf) if and only if it is accessible.
- c. Suppose given a functor $f : \mathcal{C} \rightarrow \mathcal{D}$. We write $j_f : \mathcal{D} \rightarrow \text{Fun}(\mathcal{C}^{\text{op}}, \text{Spc})$ for the composition of $j_{\mathcal{D}}$ with restriction along f and we write $j_f^{\vee} : \mathcal{D}^{\text{op}} \rightarrow \text{Fun}(\mathcal{C}, \text{Spc})$ for the composition of $j_{\mathcal{D}}^{\vee}$ with restriction along f^{op} ; we refer to these as *restricted Yoneda functors*. We will sometimes write $j_{\mathcal{D}|\mathcal{C}}$ in place of j_f , or $j_{\mathcal{D}|\mathcal{C}}^{\vee}$ in place of $j_f^{\vee}$, when the functor f may be understood from context, for example when f is the inclusion of a full subcategory.

For $X \in \text{Fun}(\mathcal{C}^{\text{op}}, \text{Spc})$, we write $L_f(X) \in \mathcal{D}$ for a left adjoint of the restricted Yoneda functor $j_f : \mathcal{D} \rightarrow \text{Fun}(\mathcal{C}^{\text{op}}, \text{Spc})$ at X , if it exists; just as we sometimes write $j_{\mathcal{D}|\mathcal{C}}$ in place of j_f , we may write $L_{\mathcal{D}|\mathcal{C}}$ in place of L_f . Dually, for $X \in \text{Fun}(\mathcal{C}, \text{Spc})$, we write $\Gamma_f(X) \in \mathcal{D}$ for a right adjoint of the restricted Yoneda functor $j_f^{\vee} : \mathcal{D} \rightarrow \text{Fun}(\mathcal{C}, \text{Spc})^{\text{op}}$ at X , if it exists, and we may write $\Gamma_{\mathcal{D}|\mathcal{C}}$ in place of Γ_f .

If $\mathcal{D}$ is cocomplete, then L_f is defined on $\mathcal{P}(\mathcal{C})$: the functor $L_f : \mathcal{P}(\mathcal{C}) \rightarrow \mathcal{D}$ is the left Kan extension of f along $j_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{P}(\mathcal{C})$, and this is the unique colimit preserving extension of f along $j_{\mathcal{C}}$. Dually, if $\mathcal{D}$ is complete, Γ_f is defined on $\mathcal{P}^{\vee}(\mathcal{C})$, the functor $\Gamma_f : \mathcal{P}^{\vee}(\mathcal{C})^{\text{op}} \rightarrow \mathcal{D}$ being the right Kan extension of f along $j_{\mathcal{C}}^{\vee} : \mathcal{C} \rightarrow \mathcal{P}^{\vee}(\mathcal{C})^{\text{op}}$, which is the unique limit preserving extension of f along $j_{\mathcal{C}}^{\vee}$.

- d. Assume that $\mathcal{C}$ is essentially small and admits finite coproducts. Then we write $\mathcal{P}_{\Sigma}(\mathcal{C})$ for the full subcategory of $\mathcal{P}(\mathcal{C}) = \text{Fun}(\mathcal{C}^{\text{op}}, \text{Spc})$ spanned by those presheaves that preserve finite products. This is a Bousfield localization of $\mathcal{P}(\mathcal{C})$: we will sometimes write $L_{\Sigma} : \mathcal{P}(\mathcal{C}) \rightarrow \mathcal{P}_{\Sigma}(\mathcal{C})$ for the left adjoint to the inclusion functor. The Yoneda embedding $j_{\mathcal{C}}$ factors through $\mathcal{P}_{\Sigma}(\mathcal{C})$. Given a functor $f : \mathcal{C} \rightarrow \mathcal{D}$ where $\mathcal{D}$ admits sifted colimits, f admits a left Kan extension $L_{f, \Sigma} : \mathcal{P}_{\Sigma}(\mathcal{C}) \rightarrow \mathcal{D}$ along $j_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{P}_{\Sigma}(\mathcal{C})$, which is the unique sifted colimit preserving extension of f along that embedding; if $\mathcal{D}$ is in fact cocomplete, then $L_{f, \Sigma}$ is the restriction of the functor $L_f : \mathcal{P}(\mathcal{C}) \rightarrow \mathcal{D}$ of b, and $L_{f, \Sigma}$ is colimit preserving if and only if f preserves finite coproducts, in which case we have $L_f \simeq L_{f, \Sigma} \circ L_{\Sigma}$.

Dually, if $\mathcal{C}$ is essentially small and admits finite products, we let $\mathcal{P}_{\Sigma}^{\vee}(\mathcal{C})$ denote the full subcategory of $\mathcal{P}^{\vee}(\mathcal{C}) = \text{Fun}(\mathcal{C}, \text{Spc})$ spanned by those copresheaves that preserve finite products.

11. Let $\mathcal{C}$ be an ∞ -category. By a *detecting subcategory* of $\mathcal{C}$ we mean a full subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$ such that the restricted Yoneda functor $j_{\mathcal{C}|\mathcal{C}_0} : \mathcal{C} \rightarrow \text{Fun}(\mathcal{C}_0^{\text{op}}, \text{Spc})$ is conservative. By a *detecting set of objects* of $\mathcal{C}$ we mean a set of objects of $\mathcal{C}$ that span a detecting subcategory.
12. For $\mathcal{C}$ an ∞ -category and I a set, we may consider the functor ∞ -category $\mathcal{C}^I = \text{Fun}(I, \mathcal{C})$. We will generally denote objects of this ∞ -category as I -indexed collections $\{C_i\}_{i \in I}$ of objects of $\mathcal{C}$. If $\mathcal{C}$ admits an initial object 0 , then for any $i \in I$ and $C \in \mathcal{C}$, we will write

$$C(-i) \in \mathcal{C}^I$$

for the object defined by $C(-i)_i = C$ and $C(-i)_j = 0$ for $j \in I \setminus \{i\}$, which corepresents the functor $\text{Map}_{\mathcal{C}}(C, (-)_i) : \mathcal{C}^I \rightarrow \text{Spc}$. The minus sign is pure notation (I is merely a set), which we use to match standard conventions in the case where $I = \mathbb{Z}$ and $\mathcal{C}$ carries a symmetric monoidal structure.

Ch. 1. Categorical preliminaries

§1.1. Partial adjoints. It will be convenient for us to be able to invoke the basic properties of adjoint functors in situations where they are defined only partially. Unaware of a suitable reference, we'll work through the relevant definitions and results here. We'll only discuss the case of right adjoints, but of course there is a formally dual picture for left adjoints. Throughout the section, we let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a functor of ∞ -categories.

Definition 1.1.1. Let $Y \in \mathcal{Y}$. A *right adjoint* of f at Y is an object $Y^\dagger \in \mathcal{X}$ together with a natural equivalence

$$\alpha : \mathrm{Map}_{\mathcal{X}}(X, Y^\dagger) \simeq \mathrm{Map}_{\mathcal{Y}}(f(X), Y)$$

for $X \in \mathcal{X}$. Given such, we refer to maps $\phi : X \rightarrow Y^\dagger$ and $\psi : f(X) \rightarrow Y$ that correspond under α as the *adjuncts* of one another.

Remark 1.1.2. Consider the restricted Yoneda functor

$$j_f : \mathcal{Y} \rightarrow \mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathrm{SPC}), \quad Y \mapsto \mathrm{Map}_{\mathcal{Y}}(f(-), Y).$$

For $Y \in \mathcal{Y}$, the existence of a right adjoint of f at Y is equivalent to $j_f(Y)$ being a representable functor, i.e. lying in the essential image of the Yoneda embedding $j_{\mathcal{X}} : \mathcal{X} \rightarrow \mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathrm{SPC})$. The Yoneda lemma implies that the right adjoint is unique when it exists.

Example 1.1.3. Suppose that f is fully faithful. Then, for any $X' \in \mathcal{X}$, the right adjoint of f at $f(X')$ is given by X' together with the natural equivalence $\mathrm{Map}_{\mathcal{X}}(X, X') \xrightarrow{\sim} \mathrm{Map}_{\mathcal{Y}}(f(X), f(X'))$ induced by f .

Definition 1.1.4. Let $\mathcal{Y}^\dagger \subseteq \mathcal{Y}$ be the full subcategory spanned by those objects Y at which f admits a right adjoint. By Remark 1.1.2, there is an essentially unique functor $u : \mathcal{Y}^\dagger \rightarrow \mathcal{X}$ making the diagram

$$\begin{array}{ccc} \mathcal{Y}^\dagger & \xrightarrow{u} & \mathcal{X} \\ \downarrow & & \downarrow j_{\mathcal{X}} \\ \mathcal{Y} & \xrightarrow{j_f} & \mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathrm{SPC}) \end{array}$$

commute, meaning that we have a natural equivalence

$$\alpha : \mathrm{Map}_{\mathcal{X}}(X, u(Y)) \simeq \mathrm{Map}_{\mathcal{Y}}(f(X), Y)$$

for $X \in \mathcal{X}$ and $Y \in \mathcal{Y}^\dagger$. We refer to the functor u as the *right adjoint* of f .

Let $\mathcal{Y}' \subseteq \mathcal{Y}$ be a full subcategory. By a *partial right adjoint* of f on $\mathcal{Y}'$ we mean a functor $u' : \mathcal{Y}' \rightarrow \mathcal{X}$ together with an equivalence $j_{\mathcal{X}} \circ u' \simeq j_f|_{\mathcal{Y}'}$; in other words, this means that $\mathcal{Y}' \subseteq \mathcal{Y}^\dagger$, i.e. the right adjoint u is defined on $\mathcal{Y}'$, and we have an equivalence $u' \simeq u|_{\mathcal{Y}'}$.

Remark 1.1.5. The language of Definition 1.1.4 grants a right adjoint to any functor of ∞ -categories. Elsewhere in the paper, we'll still use the phrase that f *admits a right adjoint* with the standard, nontrivial meaning, i.e. that $\mathcal{Y}^\dagger = \mathcal{Y}$ in the notation above.

Remark 1.1.6. Let $u : \mathcal{Y}^\dagger \rightarrow \mathcal{X}$ be the right adjoint of f . Let $y : \mathcal{I} \rightarrow \mathcal{Y}^\dagger$ be a functor whose composition with the inclusion $\mathcal{Y}^\dagger \hookrightarrow \mathcal{Y}$ admits a limit $Y \in \mathcal{Y}$. Then $Y \in \mathcal{Y}^\dagger$, so that Y is also the limit of y itself, and furthermore u preserves this limit. This follows from the fact that the restricted Yoneda functor $j_f : \mathcal{Y} \rightarrow \mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathrm{SPC})$ preserves limits and the Yoneda embedding $j_{\mathcal{X}} : \mathcal{X} \hookrightarrow \mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathrm{SPC})$ creates limits.

Definition 1.1.7. Let $u : \mathcal{Y}^\dagger \rightarrow \mathcal{X}$ be the right adjoint of f . For $Y \in \mathcal{Y}^\dagger$, we define the *counit map* $\epsilon : fu(Y) \rightarrow Y$ in $\mathcal{Y}$ to be the adjunct of $\mathrm{id}_{u(Y)}$. For $X \in \mathcal{X}$ such that $f(X) \in \mathcal{Y}^\dagger$, we define the *unit map* $\eta : X \rightarrow uf(X)$ in $\mathcal{X}$ to be the adjunct of $\mathrm{id}_{f(X)}$.

Remark 1.1.8. The counit and unit maps of Definition 1.1.7 canonically promote to natural transformations. To justify this, we review an alternative characterization of the right adjoint $u : \mathcal{Y}^\dagger \rightarrow \mathcal{X}$.

The functor $f : \mathcal{X} \rightarrow \mathcal{Y}$ corresponds to a cocartesian fibration $p : \mathcal{Z} \rightarrow \Delta^1$ with identifications $p^{-1}(0) \simeq \mathcal{X}$ and $p^{-1}(1) \simeq \mathcal{Y}$. We recall that there is a canonical functor $\ell_{\mathcal{Y}} : \mathcal{Z} \rightarrow \mathcal{Y}$ together with a natural transformation

$\beta_y : \text{id}_Z \rightarrow \ell_y$ such that, for any $Z \in \mathcal{Z}$, the map $\beta_y : Z \rightarrow \ell_y(Z)$ in $\mathcal{Z}$ is p -cocartesian (for $Z \in \mathcal{Y}$, this means that β_y is an equivalence); this is constructed as follows. Let $\text{Fun}^{\text{cocart}}(\Delta^1, \mathcal{Z})$ denote the full subcategory of $\text{Fun}(\Delta^1, \mathcal{Z})$ spanned by the p -cocartesian maps in $\mathcal{Z}$. Since p is a cocartesian fibration, we have a pullback diagram

$$\begin{array}{ccc} \text{Fun}^{\text{cocart}}(\Delta^1, \mathcal{Z}) & \xrightarrow{e_0} & \mathcal{Z} \\ p \downarrow & & \downarrow p \\ \text{Fun}(\Delta^1, \Delta^1) & \xrightarrow{e_0} & \Delta^1, \end{array}$$

where e_0 denotes evaluation at $0 \in \Delta^1$. It follows that there is a unique functor $m_y : \mathcal{Z} \times \Delta^1 \rightarrow \mathcal{Z}$ whose composition with p is the projection functor $\mathcal{Z} \times \Delta^1 \rightarrow \Delta^1$, whose restriction to $\mathcal{Z} \times \{0\}$ is id_Z , and whose restriction to $\{Z\} \times \Delta^1$, for any $Z \in \mathcal{Z}$, is a p -cocartesian map in $\mathcal{Z}$. Taking ℓ_y to be the restriction of m_y to $\mathcal{Z} \times \{1\}$, this defines the desired data.

The restriction of ℓ_y to $\mathcal{X}$ identifies with f , by definition of the correspondence between f and p , and the natural transformation β_y induces the commutativity of the following triangle:

$$\begin{array}{ccc} \mathcal{Y} & & \\ \downarrow & \searrow j_f & \\ \mathcal{Z} & \xrightarrow{j_{\mathcal{Z}|\mathcal{X}}} & \text{Fun}(\mathcal{X}^{\text{op}}, \text{Spc}). \end{array}$$

Now let $\mathcal{Z}' \subseteq \mathcal{Z}$ be the full subcategory spanned by those objects Z for which there exists a p -cartesian lift of the unique map $0 \rightarrow p(Z)$ in Δ^1 . By reasoning similar to above, there is a canonical functor $\ell_x : \mathcal{Z}' \rightarrow \mathcal{X}$ together with a transformation $\beta_x : \ell_x \rightarrow \text{id}_Z$ such that, for any $Z \in \mathcal{Z}'$, the map $\beta_x : \ell_x(Z) \rightarrow Z$ is p -cartesian. Unwinding definitions, we find that $\mathcal{X} \subseteq \mathcal{Z}'$ for trivial reasons and that $\mathcal{Z}' \cap \mathcal{Y} = \mathcal{Y}^\dagger$, with β_x inducing the commutativity of the triangle

$$\begin{array}{ccc} & \ell_x & \mathcal{X} \\ & \nearrow & \downarrow j_x \\ \mathcal{Z} & \xrightarrow{j_{\mathcal{Z}|\mathcal{X}}} & \text{Fun}(\mathcal{X}^{\text{op}}, \text{Spc}), \end{array}$$

which, in combination with the previous commutative triangle, identifies $\ell_x|_{\mathcal{Y}^\dagger}$ with the right adjoint $u : \mathcal{Y}^\dagger \rightarrow \mathcal{X}$. Under this identification, the counit map $\epsilon : fu(Y) \rightarrow Y$ for $Y \in \mathcal{Y}^\dagger$ is given by the composition

$$fu(Y) \simeq \ell_y \ell_x(Y) \xrightarrow{\beta_x} \ell_y(Y) \xrightarrow{\beta_y} Y,$$

and the unit map $\eta : X \rightarrow uf(X)$ for $X \in \mathcal{X}$ such that $f(X) \in \mathcal{Y}^\dagger$ is given by the composition

$$X \xrightarrow{\beta_x} \ell_x(X) \xrightarrow{\ell_x(\beta_y)} \ell_x \ell_y(X) \simeq uf(X).$$

Next we review an interaction between adjoints and full faithfulness.

Proposition 1.1.9. *Let $u : \mathcal{Y}^\dagger \rightarrow \mathcal{X}$ be the right adjoint of f . Let $\mathcal{X}^\dagger \subseteq \mathcal{X}$ be the full subcategory spanned by those objects X for which $f(X) \in \mathcal{Y}^\dagger$, and let $\mathcal{X}^\ddagger \subseteq \mathcal{X}'$ be the full subcategory spanned by those objects X for which the unit map $\eta : X \rightarrow uf(X)$ is an equivalence. Then the following statements hold:*

1. *Let $X \in \mathcal{X}^\dagger$. Then $X \in \mathcal{X}^\ddagger$ if and only if, for every $X' \in \mathcal{X}$, the map of spaces*

$$\text{Map}_{\mathcal{X}}(X', X) \rightarrow \text{Map}_{\mathcal{Y}}(f(X'), f(X))$$

induced by f is an equivalence.

2. *The restriction of f to $\mathcal{X}^\ddagger$ is fully faithful.*
3. *We have $\mathcal{X}^\ddagger = \mathcal{X}$ if and only if f is fully faithful.*
4. *$\mathcal{X}^\ddagger$ is closed under limits in $\mathcal{X}$ that are preserved by f .*

Proof. 1 follows from the fact that, for any $X \in \mathcal{X}^\dagger$ and $X' \in \mathcal{X}$, the composition

$$\text{Map}_{\mathcal{X}}(X', X) \xrightarrow{f} \text{Map}_{\mathcal{Y}}(f(X'), f(X)) \simeq \text{Map}_{\mathcal{X}}(X', uf(X))$$

is given by composition with $\eta : X \rightarrow uf(X)$. 2 follows immediately from 1. The “only if” direction of 3 follows from 2, and the “if” direction follows from Example 1.1.3. 4 follows from Remark 1.1.6. $\square$

Lastly, we will need the following observation about adjoints of a limit of a diagram of functors.

Proposition 1.1.10. *Suppose that $f : \mathcal{X} \rightarrow \mathcal{Y}$ is the limit of a diagram of functors $\{f_i : \mathcal{X}_i \rightarrow \mathcal{Y}_i\}_{i \in \mathcal{I}}$; for each $i \in \mathcal{I}$, let $p_i : \mathcal{X} \rightarrow \mathcal{X}_i$ and $q_i : \mathcal{Y} \rightarrow \mathcal{Y}_i$ denote the projection maps. Let $Y \in \mathcal{Y}$, and suppose that the following conditions hold:*

1. *For each $i \in \mathcal{I}$, the right adjoint $q_i(Y)^\dagger$ of f_i at $q_i(Y)$ exists.*
2. *For each map $\phi : i \rightarrow j$ in $\mathcal{I}$, inducing a commutative diagram*

$$\begin{array}{ccc} \mathcal{X}_i & \xrightarrow{f_i} & \mathcal{Y}_i \\ p_\phi \downarrow & & \downarrow q_\phi \\ \mathcal{X}_j & \xrightarrow{f_j} & \mathcal{Y}_j, \end{array}$$

the canonical map $p_\phi(q_i(Y)^\dagger) \rightarrow q_j(Y)^\dagger$ is an equivalence.

Then the right adjoint $Y^\dagger$ of f at Y exists, and for each $i \in \mathcal{I}$, the canonical map $p_i(Y^\dagger) \rightarrow q_i(Y)^\dagger$ is an equivalence.

Lemma 1.1.11. *Suppose given a commutative diagram of ∞ -categories*

$$\begin{array}{ccc} \mathcal{X}_+ & \xrightarrow{f_+} & \mathcal{Y}_+ \\ & \searrow s & \swarrow t \\ & \mathcal{J} & \end{array}$$

in which s and t are locally cocartesian fibrations and f_+ carries locally s -cocartesian maps to locally t -cocartesian maps. Let $i \in \mathcal{J}$, let $f_i : \mathcal{X}_i \rightarrow \mathcal{Y}_i$ denote the fiber of f_+ over i , let $Y_i \in \mathcal{Y}_i$, and let $Y_i^\dagger \in \mathcal{X}_i$ be a right adjoint of f_i at Y_i . Then $Y_i^\dagger$ is also a right adjoint of f_+ at Y_i .

Proof. Let $\epsilon : f_+(Y_i^\dagger) = f_i(Y_i^\dagger) \rightarrow Y_i$ denote the counit map in $\mathcal{Y}_i$. We would like to show that the composition

$$\mathrm{Map}_{\mathcal{X}_+}(X, Y_i^\dagger) \xrightarrow{f_+} \mathrm{Map}_{\mathcal{Y}_+}(f_+(X), f_+(Y_i^\dagger)) \xrightarrow{\epsilon} \mathrm{Map}_{\mathcal{Y}_+}(f_+(X), Y_i)$$

is an equivalence for any $X \in \mathcal{X}$. Each map above lies over $\mathrm{Map}_{\mathcal{J}}(s(X), i)$; let us fix a map $\phi : s(X) \rightarrow i$ in $\mathcal{J}$ and show that the induced composition on fibers over ϕ is an equivalence. By hypothesis, we may choose a locally s -cocartesian map $X \rightarrow X'$ over ϕ , and then the induced map $f_+(X) \rightarrow f_+(X') = f_i(X')$ is locally t -cocartesian. Then our induced composition on fibers over ϕ identifies with the composition

$$\mathrm{Map}_{\mathcal{X}_i}(X', Y_i^\dagger) \xrightarrow{f_i} \mathrm{Map}_{\mathcal{Y}_i}(f_i(X'), f_i(Y_i^\dagger)) \xrightarrow{\epsilon} \mathrm{Map}_{\mathcal{Y}_i}(f_i(X'), Y_i)$$

which we know to be an equivalence. $\square$

Proof of Proposition 1.1.10. Unstraightening the given diagram of functors produces a commutative diagram of ∞ -categories

$$\begin{array}{ccc} \mathcal{X}_+ & \xrightarrow{f_+} & \mathcal{Y}_+ \\ & \searrow s & \swarrow t \\ & \mathcal{J} & \end{array}$$

where s and t are cocartesian fibrations with identifications $s^{-1}(i) \simeq \mathcal{X}_i$ and $t^{-1}(i) \simeq \mathcal{Y}_i$ for each $i \in \mathcal{J}$, and f_+ carries s -cocartesian maps to t -cocartesian maps. We may identify $\mathcal{X}$ with the full subcategory of $\mathrm{Fun}_{\mathcal{J}}(\mathcal{J}, \mathcal{X}_+)$ spanned by those sections of s that carry each map to an s -cocartesian map, and similarly for $\mathcal{Y}$, and under these identifications, the functor $f : \mathcal{X} \rightarrow \mathcal{Y}$ is given by composition with f_+ .

Let $\mathcal{Y}'_+ \subseteq \mathcal{Y}_+$ be the full subcategory of spanned by the objects $q_i(Y)$ for $i \in \mathcal{I}$. By Lemma 1.1.11, f_+ admits a partial right adjoint u_+ on $\mathcal{Y}'_+$, carrying $q_i(Y)$ to $q_i(Y)^\dagger$. Regarding $Y \in \mathcal{Y}$ as a section of t , we obtain a section $Y^\dagger := u_+ \circ Y$ of s , and by hypothesis 2, $Y^\dagger$ in fact lies in $\mathcal{X}$. Using the naturality of the counit map $\epsilon : f_+ u_+(Y') \rightarrow Y'$ for $Y' \in \mathcal{Y}'_+$ (Remark 1.1.8), we have an induced map $\epsilon : f(Y^\dagger) = f_+ \circ Y^\dagger \rightarrow Y^\dagger$.

To finish the proof, it suffices to show that the composite map

$$\mathrm{Map}_{\mathcal{X}}(X, Y^\dagger) \xrightarrow{f} \mathrm{Map}_{\mathcal{Y}}(f(X), f(Y^\dagger)) \xrightarrow{\epsilon} \mathrm{Map}_{\mathcal{Y}}(f(X), Y)$$

is an equivalence for any $X \in \mathcal{X}$. This follows from the fact that it identifies with a limit over $i \in \mathcal{I}$ of the composition

$$\begin{aligned} \mathrm{Map}_{\mathcal{X}_i}(p_i(X), p_i(Y^\dagger)) &\simeq \mathrm{Map}_{\mathcal{X}_i}(p_i(X), q_i(Y)^\dagger) \\ &\xrightarrow{f_i} \mathrm{Map}_{\mathcal{Y}_i}(f_i(p_i(X)), f_i(q_i(Y)^\dagger)) \\ &\xrightarrow{\epsilon} \mathrm{Map}_{\mathcal{Y}_i}(f_i(p_i(X)), q_i(Y)) \\ &\simeq \mathrm{Map}_{\mathcal{Y}_i}(q_i(f(X)), q_i(Y)), \end{aligned}$$

which is an equivalence. $\square$

§1.2. Density. At the core of this paper is the following ubiquitous phenomenon: an object of some ∞ -category $\mathcal{X}$ that we are studying may be built in a canonical fashion from the objects of some other ∞ -category, for instance some full subcategory of $\mathcal{X}$, which perhaps we know better. There are two dual versions of this: the building can be by colimits or by limits; our main constructions will involve both at once.

The phenomenon goes by the name of *density*. In this section, we'll review the relevant definitions and a useful mechanism for detecting it. We fix a functor of ∞ -categories $u : \mathcal{Y} \rightarrow \mathcal{X}$ for the discussion, and we begin by repeating and elaborating on a point from Convention 0.3.10.c (which involves the notions of §1.1).

Notation 1.2.1. We write $L_u : \mathrm{Fun}(\mathcal{Y}^{\mathrm{op}}, \mathrm{SPC})^\dagger \rightarrow \mathcal{X}$ for the left adjoint of the restricted Yoneda functor $j_u : \mathcal{X} \rightarrow \mathrm{Fun}(\mathcal{Y}^{\mathrm{op}}, \mathrm{SPC})$ and we write $\Gamma_u : (\mathrm{Fun}(\mathcal{Y}, \mathrm{SPC})^\dagger)^{\mathrm{op}} \rightarrow \mathcal{X}$ for the right adjoint of $j_u^\vee : \mathcal{X} \rightarrow \mathrm{Fun}(\mathcal{Y}, \mathrm{SPC})^{\mathrm{op}}$ (the adjoints being defined partially as in Definition 1.1.4).

Proposition 1.2.2. *The functors L_u and Γ_u of Notation 1.2.1 may be alternatively characterized as follows:*

1. *A functor $\psi : \mathcal{Y}^{\mathrm{op}} \rightarrow \mathrm{SPC}$ is in the domain of $L_u : \mathrm{Fun}(\mathcal{Y}^{\mathrm{op}}, \mathrm{SPC})^\dagger \rightarrow \mathrm{SPC}$ if and only if the left Kan extension of $u : \mathcal{Y} \rightarrow \mathcal{X}$ along the Yoneda embedding $j_{\mathcal{Y}} : \mathcal{Y} \rightarrow \mathrm{Fun}(\mathcal{Y}^{\mathrm{op}}, \mathrm{SPC})$ is defined at ψ , i.e. the composition*

$$\mathcal{Y}_{/\psi} = \mathcal{Y} \times_{\mathrm{Fun}(\mathcal{Y}^{\mathrm{op}}, \mathrm{SPC})} \mathrm{Fun}(\mathcal{Y}^{\mathrm{op}}, \mathrm{SPC})_{/\psi} \xrightarrow{p} \mathcal{Y} \xrightarrow{u} \mathcal{X},$$

where p denotes the projection functor, admits a colimit; in particular, all representable functors are in the domain of L_u .

Dually, a functor $\psi : \mathcal{Y} \rightarrow \mathrm{SPC}$ is in the domain of $\Gamma_u : (\mathrm{Fun}(\mathcal{Y}, \mathrm{SPC})^\dagger)^{\mathrm{op}} \rightarrow \mathrm{SPC}$ if and only if the right Kan extension of $u : \mathcal{Y} \rightarrow \mathcal{X}$ along $j_{\mathcal{Y}}^\vee : \mathcal{Y} \rightarrow \mathrm{Fun}(\mathcal{Y}, \mathrm{SPC})^{\mathrm{op}}$ is defined at ψ , and in particular, all corepresentable functors are in the domain of Γ_u .

2. *$L_u : \mathrm{Fun}(\mathcal{Y}^{\mathrm{op}}, \mathrm{SPC})^\dagger \rightarrow \mathcal{X}$ is canonically equivalent to the left Kan extension of u along $j_{\mathcal{Y}}$. Dually, $\Gamma_u : (\mathrm{Fun}(\mathcal{Y}, \mathrm{SPC})^\dagger)^{\mathrm{op}} \rightarrow \mathcal{X}$ is canonically equivalent to the right Kan extension of u along $j_{\mathcal{Y}}^\vee$.*

Lemma 1.2.3. *The square*

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{u} & \mathcal{X} \\ j_{\mathcal{Y}} \downarrow & & \downarrow j_{\mathcal{X}}^\vee \\ \mathrm{Fun}(\mathcal{Y}^{\mathrm{op}}, \mathrm{SPC}) & \xrightarrow{j_{j_u}^\vee} & \mathrm{Fun}(\mathcal{X}, \mathrm{SPC})^{\mathrm{op}} \end{array}$$

commutes, and the lower horizontal arrow is left Kan extended along the left vertical arrow.

Proof. The Yoneda lemma gives a natural equivalence

$$j_{j_u}^\vee(j_{\mathcal{Y}}(Y))(X) = \text{Map}_{\text{Fun}(\mathcal{Y}^{\text{op}}, \text{Spc})}(j_{\mathcal{Y}}(Y), j_u(X)) \simeq j_u(X)(Y) = \text{Map}_{\mathcal{X}}(u(Y), X) = j_{\mathcal{X}}^\vee(u(Y))(X)$$

for $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$, establishing the commutativity of the square. For the Kan extension assertion, it's convenient to take opposites (we'll maintain the same notation for the functors, however); letting $e_X : \text{Fun}(\mathcal{X}, \text{Spc}) \rightarrow \text{Spc}$ denote evaluation at an object $X \in \mathcal{X}$, and letting

$$e : \text{Fun}(\mathcal{Y}^{\text{op}}, \text{Spc})^{\text{op}} \rightarrow \text{Spc}$$

denote the right Kan extension of $e_X \circ j_{j_u}^\vee \circ j_{\mathcal{Y}} : \mathcal{Y}^{\text{op}} \rightarrow \text{Spc}$ along $j_{\mathcal{Y}} : \mathcal{Y}^{\text{op}} \rightarrow \text{Fun}(\mathcal{Y}^{\text{op}}, \text{Spc})^{\text{op}}$, we would like to show that the canonical map $e_X \circ j_{j_u}^\vee \rightarrow e$ is an equivalence. By the Yoneda lemma, this is equivalent to the induced map

$$\text{Map}_{\text{Fun}(\text{Fun}(\mathcal{Y}^{\text{op}}, \text{Spc})^{\text{op}}, \text{Spc})}(j_{\text{Fun}(\mathcal{Y}^{\text{op}}, \text{Spc})}(\psi), e_X \circ j_{j_u}^\vee) \rightarrow \text{Map}_{\text{Fun}(\text{Fun}(\mathcal{Y}^{\text{op}}, \text{Spc})^{\text{op}}, \text{Spc})}(j_{\text{Fun}(\mathcal{Y}^{\text{op}}, \text{Spc})}(\psi), e)$$

being an equivalence for any $\psi \in \text{Fun}(\mathcal{Y}^{\text{op}}, \text{Spc})$. By the adjunction between right Kan extension and restriction, this is furthermore equivalent to the map

$$\text{Map}_{\text{Fun}(\text{Fun}(\mathcal{Y}^{\text{op}}, \text{Spc})^{\text{op}}, \text{Spc})}(j_{\text{Fun}(\mathcal{Y}^{\text{op}}, \text{Spc})}(\psi), e_X \circ j_{j_u}^\vee) \rightarrow \text{Map}_{\text{Fun}(\mathcal{Y}^{\text{op}}, \text{Spc})}(j_{\mathcal{Y}}^*(j_{\text{Fun}(\mathcal{Y}^{\text{op}}, \text{Spc})}^\vee(\psi)), j_{\mathcal{Y}}^*(e_X \circ j_{j_u}^\vee))$$

given by restriction along $j_{\mathcal{Y}} : \mathcal{Y}^{\text{op}} \rightarrow \text{Fun}(\mathcal{Y}^{\text{op}}, \text{Spc})^{\text{op}}$ being an equivalence. This follows from identifying

$$e_X \circ j_{j_u}^\vee \simeq j_{\text{Fun}(\mathcal{Y}^{\text{op}}, \text{Spc})}(j_u(X))$$

and then using the two facts, again from the Yoneda lemma, that $j_{\mathcal{Y}}^* \circ j_{\text{Fun}(\mathcal{Y}^{\text{op}}, \text{Spc})} \simeq \text{id}_{\text{Fun}(\mathcal{Y}^{\text{op}}, \text{Spc})}$ and that $j_{\text{Fun}(\mathcal{Y}^{\text{op}}, \text{Spc})}$ is fully faithful. $\square$

Proof of Proposition 1.2.2. We treat the case of L_u . By its definition as the left adjoint of j_u , it is characterized by an equivalence $j_{\mathcal{X}}^\vee \circ L_u \simeq j_{j_u}^\vee$, and a functor $\psi : \mathcal{Y}^{\text{op}} \rightarrow \text{Spc}$ is in its domain if and only if $j_{j_u}^\vee(\psi) : \mathcal{X} \rightarrow \text{Spc}$ is corepresentable functor. The two statements now follow from Lemma 1.2.3 and the fact that the Yoneda embedding $j_{\mathcal{X}}^\vee : \mathcal{X} \rightarrow \text{Fun}(\mathcal{X}, \text{Spc})^{\text{op}}$ creates colimits. $\square$

Remark 1.2.4. It follows from Proposition 1.2.2 that, if $\mathcal{X}$ admits small colimits, then L_u is defined on $\mathcal{P}(\mathcal{Y}) \subseteq \text{Fun}(\mathcal{Y}^{\text{op}}, \text{Spc})$ (Convention 0.3.10.b), and its restriction there is the unique extension of u along $j_{\mathcal{Y}}$ that preserves small colimits. Dually, if $\mathcal{X}$ admits small limits, then Γ_u is defined on $\mathcal{P}^\vee(\mathcal{Y}) \subseteq \text{Fun}(\mathcal{Y}, \text{Spc})$, and its restriction there is the unique extension of u along $j_u^\vee$ that preserves small limits.

We now formulate the central notions of this section.

Definition 1.2.5. We say that an object $X \in \mathcal{X}$ is *u-proximal* if $j_u(X)$ is in the domain of L_u , and *u-realized* if it is *u-proximal* and the counit map $\epsilon : L_u j_u(X) \rightarrow X$ is an equivalence; dually, we say that X is *u-coproximal* if $j_u^\vee(X)$ is in the domain of Γ_u , and *u-corealized* if it is *u-coproximal* and the unit map $\eta : X \rightarrow \Gamma_u j_u^\vee(X)$ is an equivalence. In situations where the functor u may be understood from context—for instance, when it is the inclusion of a subcategory—we may replace u with $\mathcal{Y}$ in this terminology.

We say that the functor $u : \mathcal{Y} \rightarrow \mathcal{X}$ is *dense* (resp. *codense*) if every object of $\mathcal{X}$ is *u-realized* (resp. *u-corealized*). When u is the inclusion of a subcategory, we may instead say that $\mathcal{Y}$ is a dense (resp. codense) subcategory of $\mathcal{X}$ when this holds.

Remark 1.2.6. It follows from Proposition 1.2.2 that the definition of density above is equivalent to that of [28, Tag 03VP].

Example 1.2.7. If $\mathcal{X}$ admits small colimits (resp. small limits) and $\mathcal{Y}$ is essentially small, then every object of $\mathcal{X}$ is *u-proximal* (resp. *u-coproximal*).

General observations from the previous section give us the following relations between Definition 1.2.5 and full faithfulness of restricted Yoneda functors:

Proposition 1.2.8. *The restriction of $j_u : \mathcal{X} \rightarrow \text{Fun}(\mathcal{Y}^{\text{op}}, \text{Spc})$ to the full subcategory of $\mathcal{X}$ spanned by the *u-realized* objects is fully faithful. Dually, the restriction of $j_u^\vee : \mathcal{X} \rightarrow \text{Fun}(\mathcal{Y}, \text{Spc})^{\text{op}}$ to the full subcategory of $\mathcal{X}$ spanned by the *u-corealized* objects is fully faithful.*

Proof. This is a special case of Proposition 1.1.9.2. $\square$

Proposition 1.2.9. *The functor $u : \mathcal{Y} \rightarrow \mathcal{X}$ is dense if and only if $j_u : \mathcal{X} \rightarrow \text{Fun}(\mathcal{Y}^{\text{op}}, \text{Spc})$ is fully faithful. Dually, u is codense if and only if $j_u^\vee : \mathcal{X} \rightarrow \text{Fun}(\mathcal{Y}, \text{Spc})^{\text{op}}$ is fully faithful.*

Proof. This is a special case of Proposition 1.1.9.3. $\square$

Example 1.2.10. Suppose that $\mathcal{X}$ is presentable, that $\mathcal{Y}$ is essentially small, and that u is dense. Then we have a Bousfield localization of presentable ∞ -categories

$$\mathcal{P}(\mathcal{Y}) = \text{Fun}(\mathcal{Y}^{\text{op}}, \text{Spc}) \begin{matrix} \xrightarrow{L_u} \\ \xleftarrow{j_u} \end{matrix} \mathcal{X}.$$

The previous section also supplies us with the following closure properties of u -approximation and u -coapproximation.

Proposition 1.2.11. *The full subcategory of u -realized objects in $\mathcal{X}$ is closed under colimits that are preserved by j_u . Dually, the full subcategory of u -corealized objects in $\mathcal{X}$ is closed under limits that are preserved by $j_u^\vee$.*

Proof. This is a special case of Proposition 1.1.9.4. $\square$

Example 1.2.12. Suppose that $\mathcal{X}$ admits filtered colimits and that every object in the essential image of u is compact. Then j_u preserves filtered colimits, and so Proposition 1.2.11 implies that the full subcategory of u -realized objects in $\mathcal{X}$ is closed under filtered colimits.

The theory of hypercoverings supplies additional examples of colimits and limits preserved by restricted Yoneda functors, which will play an important role in Ch. 6. The last part of this section is devoted to reviewing how this works, largely following writings of Lurie (in particular [29, §A.5]).

Notation 1.2.13. For each nonnegative integer n , we let Δ_s^n denote the semisimplicial set represented by the object $[n] \in \Delta_s$, and we let $\partial\Delta_s^n \subseteq \Delta_s^n$ be the $(n-1)$ -skeleton of Δ_s^n (in the case $n=0$, we take this to mean that $\partial\Delta_s^n$ is empty).

Remark 1.2.14. Note that $\partial\Delta_s^n$ and Δ_s^n are finite semisimplicial sets, with finite cell decompositions: that is, they can be obtained from the empty semisimplicial set via a finite sequence of pushouts of the inclusion maps $\partial\Delta_s^k \hookrightarrow \Delta_s^k$.

Notation 1.2.15. Let $\mathcal{S}$ be an ∞ -category admitting finite limits, and let $S_\bullet : \Delta_s^{\text{op}} \rightarrow \mathcal{S}$ be a semisimplicial object of $\mathcal{S}$. Let $\mathcal{P}^{\text{so}}(\Delta_s)$ be the smallest full subcategory of $\mathcal{P}(\Delta_s)$ that contains the essential image of the left Yoneda functor $j_{\Delta_s} : \Delta_s \hookrightarrow \mathcal{P}(\Delta_s)$ and is closed under finite colimits. Then $S_\bullet$ admits a right Kan extension $S_\bullet[-] : \mathcal{P}^{\text{so}}(\Delta_s)^{\text{op}} \rightarrow \mathcal{S}$ along j_{Δ_s} . Below, we will consider the evaluation of $S_\bullet[-]$ on two classes of objects in $\mathcal{P}^{\text{so}}(\Delta_s)$: for each nonnegative integer n , we have $S_\bullet[\Delta_s^n] \simeq S_n \in \mathcal{S}$, and we also have $S_\bullet[\partial\Delta_s^n] \in \mathcal{S}$; the latter is called the n -th *matching object* of $S_\bullet$, and the inclusion $\partial\Delta_s^n \hookrightarrow \Delta_s^n$ induces a map $S_n \simeq S_\bullet[\Delta_s^n] \rightarrow S_\bullet[\partial\Delta_s^n]$, which we call the n -th *matching map* of $S_\bullet$.

Definition 1.2.16. Let $\mathcal{S}$ be an ∞ -category admitting finite limits, and let $p : T \rightarrow S$ be a map in $\mathcal{S}$. Let $T_\bullet : \Delta^{\text{op}} \rightarrow \mathcal{S}$ be the Čech nerve of p , let $\underline{S} : \Delta^{\text{op}} \rightarrow \mathcal{S}$ denote the constant diagram with value S , and let $p' : T_\bullet \rightarrow \underline{S}$ be the canonical such map. We say that p is an *effective epimorphism* if p' exhibits S as a colimit of $T_\bullet$.

Example 1.2.17. A map of spaces $T \rightarrow S$ is an effective epimorphism if and only if the induced map of sets $\pi_0(T) \rightarrow \pi_0(S)$ is a surjection.

Definition 1.2.18. Let $\mathcal{S}$ be an ∞ -category admitting finite limits. Let $S \in \mathcal{S}$ and let $S_\bullet$ be a semisimplicial object of $\mathcal{S}_{/S}$. We say that $S_\bullet$ is a *hypercovering* of S if, for each nonnegative integer n , the matching map $S_n \simeq S_\bullet[\Delta_s^n] \rightarrow S_\bullet[\partial\Delta_s^n]$ in $\mathcal{S}_{/S}$ is an effective epimorphism.

Remark 1.2.19. Some care needs to be taken in parsing Definition 1.2.18 correctly: the matching object $S_\bullet[\partial\Delta_s^n]$ is defined as in Notation 1.2.15, but in the ∞ -category $\mathcal{S}_{/S}$ rather than $\mathcal{S}$. For example, $S_\bullet[\partial\Delta_s^0] \simeq S$ and $S_\bullet[\partial\Delta_s^1] \simeq S_0 \times_S S_0$.

We may reformulate the definition purely within $\mathcal{S}$, as follows. First, we say that a map $U_\bullet \rightarrow T_\bullet$ of semisimplicial objects of $\mathcal{S}$ is a *trivial Kan fibration* if, for each nonnegative integer n , the induced map

$$U_\bullet[\Delta_s^n] \rightarrow T_\bullet[\Delta_s^n] \times_{T_\bullet[\partial\Delta_s^n]} U_\bullet[\partial\Delta_s^n]$$

is an effective epimorphism (here the construction of Notation 1.2.15 is just used in $\mathcal{S}$). Now, let $\rho : \mathcal{S}_{/S} \rightarrow \mathcal{S}$ be the forgetful functor, and let $\underline{S}$ denote the constant semisimplicial object of $\mathcal{S}$ with value S . Then we may think of our semisimplicial object $S_\bullet$ of $\mathcal{S}_{/S}$ equivalently as a map $q : \rho(S_\bullet) \rightarrow \underline{S}$ of semisimplicial objects of $\mathcal{S}$, and $S_\bullet$ is a hypercovering of S if and only if q is a trivial Kan fibration. To understand this, observe that $\underline{S}[\Delta_s^n] \simeq \underline{S}_n \simeq S$, and $S \times_{\underline{S}[\partial\Delta_s^n]} \rho(S_\bullet)[\partial\Delta_s^n] \simeq S_\bullet[\partial\Delta_s^n]$.

Theorem 1.2.20. [29, Lemma A.5.3.7] *Let S be a space and let $S_\bullet : \Delta_s^{\text{op}} \rightarrow \text{Spc}_{/S}$ be a hypercovering of S . Then the induced map $\text{colim}(S_\bullet) \rightarrow S$ is an equivalence.*

It is by virtue of Theorem 1.2.20 that hypercoverings are relevant to the discussion earlier in this section, as we now explain. We return to the setting of our functor $u : \mathcal{Y} \rightarrow \mathcal{X}$.

Definition 1.2.21. Let $X \in \mathcal{X}$.

1. We say that a map $p : X' \rightarrow X$ in $\mathcal{X}$ is a *u-covering* if, for any $Y \in \mathcal{Y}$ and map $q : u(Y) \rightarrow X$, there exists a map $q' : u(Y) \rightarrow X'$ such that $q \simeq pq'$; this is equivalent to the induced map $j_u(p) : j_u(X') \rightarrow j_u(X)$ in $\text{Fun}(\mathcal{Y}^{\text{op}}, \text{Spc})$ being an effective epimorphism (see Example 1.2.17).

Dually, we say that a map $p : X \rightarrow X'$ in $\mathcal{X}$ is a *u-cocovering* if, for any $Y \in \mathcal{Y}$ and map $q : X \rightarrow u(Y)$, there exists a map $q' : X' \rightarrow u(Y)$ such that $q \simeq q'p$; this is equivalent to the induced map $j_u^v(p) : j_u^v(X') \rightarrow j_u^v(X)$ in $\text{Fun}(\mathcal{Y}, \text{Spc})$ being an effective epimorphism.

2. A *u-hypercovering* of X is a semisimplicial object $X_\bullet : \Delta_s^{\text{op}} \rightarrow \mathcal{X}_{/X}$ such that $j_u(X_\bullet)$ is a hypercovering of $j_u(X)$ in $\text{Fun}(\mathcal{Y}^{\text{op}}, \text{Spc})$. Dually, a *u-cohypercovering* of X is a cosemisimplicial object $X^\bullet : \Delta_s \rightarrow \mathcal{X}_{/X}$ such that $j_u^v(X^\bullet)$ is a hypercovering of $j_u^v(X)$ in $\text{Fun}(\mathcal{Y}, \text{Spc})$.

As in Definition 1.2.5, we may replace u by $\mathcal{Y}$ in this terminology when appropriate.

Proposition 1.2.22. *Let $X \in \mathcal{X}$. If X admits a u-hypercovering $X_\bullet$ that exhibits X as the colimit of the underlying semisimplicial object in $\mathcal{X}$ and such that X_n is u-realized for each nonnegative integer n , then X is u-realized.*

Dually, if X admits a u-cohypercovering $X^\bullet$ that exhibits X as the limit of the underlying cosemisimplicial object in $\mathcal{X}$ and such that X^n is u-corealized for each nonnegative integer n , then X is u-corealized.

Proof. Given Theorem 1.2.20, this is a special case of Proposition 1.2.11. $\square$

The following abstract existence result, which in some situations may be applied with $\mathcal{X}'$ some full subcategory of u -realized objects and $\mathfrak{C}$ some class of u -coverings (or dually u -corealized objects and u -cocoverings) to produce u -hypercoverings by u -realized objects (or u -cohypercoverings by u -corealized objects) in $\mathcal{X}$, contributes to the utility of Proposition 1.2.22.

Proposition 1.2.23. *Let $\mathcal{X}' \subseteq \mathcal{X}$ be a full subcategory and let $\mathfrak{C}$ be a class of maps in $\mathcal{X}$. Suppose that $\mathcal{X}$ admits finite limits and that, for every $X \in \mathcal{X}$, there exists a map $X' \rightarrow X$ in $\mathfrak{C}$ where $X' \in \mathcal{X}'$. Then, for every $X \in \mathcal{X}$, there exists a semisimplicial object $X_\bullet : \Delta_s^{\text{op}} \rightarrow \mathcal{X}_{/X}$ such that, for each nonnegative integer n , we have $X_n \in \mathcal{X}'$ and the matching map $X_n \rightarrow X_\bullet[\partial\Delta_s^n]$ lies in $\mathfrak{C}$.*

Proof. This is a consequence of the fact that a semisimplicial object $X_\bullet$ may be constructed by inductively specifying each object X_n together with its matching map $X_n \rightarrow X_\bullet[\partial\Delta_s^n]$, keeping in mind that the target depends only on the restriction of $X_\bullet$ to the full subcategory of Δ_s spanned by finite linearly ordered sets of cardinality less than n [27, Remark A.2.9.16]. $\square$

Ch. 2. Algebraic structure

The central question of this chapter can be stated in heuristic terms as follows.

Question 2.0.1. *Let $\mathcal{X}$ be a presentable ∞ -category and let $u : \mathcal{Y} \rightarrow \mathcal{X}$ be a functor. What is the natural algebraic structure carried by $u(Y) \in \mathcal{X}$ for $Y \in \mathcal{Y}$?*

To make this precise, we need to fix a meaning of “algebraic structure” for objects of $\mathcal{X}$. There are different ways we might do this; indeed, much has been written about these matters, which belong to the subject of universal algebra. We will discuss two approaches here. Both stem from the ideas of Lawvere’s thesis, so let us begin there for orientation.

Most classically, universal algebra concerns algebraic structures on sets, and this notion is made precise in a syntactic fashion: a type of structure is presented by a collection of finitary operations and equational laws satisfied by these operations. In [22], Lawvere explains the following categorical perspective on this subject:

1. Such a syntax may be encoded more invariantly by what we will refer to as a *classical algebraic theory*: a locally small category $\mathcal{T}$ that admits finite products, together with an object $T \in \mathcal{T}$ that generates $\mathcal{T}$ under finite products. (Maps $T^{\times n} \rightarrow T$ in $\mathcal{T}$ correspond to n -ary operations, and the equational laws are encoded by composition in $\mathcal{T}$.)
2. For any classical algebraic theory $\mathcal{T}$, its associated semantics may be described as follows: sets equipped with the structure specified by $\mathcal{T}$, which we will refer to as *classical models* of $\mathcal{T}$, assemble into a category $\text{Mod}(\mathcal{T})$, which identifies with the category $\text{Fun}'(\mathcal{T}, \text{Set})$ of finite product-preserving functors from $\mathcal{T}$ to Set ; and under this identification, the “underlying set” functor $\text{Mod}(\mathcal{T}) \rightarrow \text{Set}$ is given by evaluation at the generator T .
3. Given any category $\mathcal{Y}$ equipped with a functor $u : \mathcal{Y} \rightarrow \text{Set}$, we may consider the full subcategory of the functor category $\text{Fun}(\mathcal{Y}, \text{Set})$ generated under finite products by u ; let us denote this by $\text{Str}(\mathcal{Y})$ (though this notation is of course abusive: $\text{Str}(\mathcal{Y})$ depends on u). Under the mild hypothesis that $\text{Str}(\mathcal{Y})$ be locally small, this defines a classical algebraic theory.
4. The constructions of 2 and 3 are functorial and related by a reflective adjunction:
 - a. For any classical algebraic theory $\mathcal{T}$, there is a natural equivalence $\mathcal{T} \xrightarrow{\sim} \text{Str}(\text{Mod}(\mathcal{T}))$. In this way, a classical algebraic theory may be recovered from its semantics.
 - b. Any functor $u : \mathcal{Y} \rightarrow \text{Set}$ satisfying the size hypothesis from 3 factors through a canonical functor $\eta : \mathcal{Y} \rightarrow \text{Mod}(\text{Str}(\mathcal{Y}))$; in other words, for $Y \in \mathcal{Y}$, the set $u(Y)$ is naturally a classical model of $\text{Str}(\mathcal{Y})$. In the case that $\mathcal{Y} = \text{Mod}(\mathcal{T})$ for a classical algebraic theory $\mathcal{T}$, the functor η corresponds to the identity functor under the identification of a, in particular is an equivalence. Furthermore, η is an equivalence if and only if u is algebraic in the sense of Definition 0.1.3: that is, it admits a left adjoint, is conservative, and creates sifted¹ colimits.

Thus, we have two equivalent categorical formulations of algebraic structure on sets: classical algebraic theories as in 1 and algebraic categories over Set as in 4. Moreover, the former point of view includes a very explicit answer to Question 2.0.1 (in the case $\mathcal{X} = \text{Set}$): namely, the construction Str of 3.

Our goal in this chapter is to lay out an analogous picture for a general presentable ∞ -category $\mathcal{X}$. In §2.2 (after a bit of preparation in §2.1), we’ll review a generalized notion of theories in this context, the classical form of which goes back to work of Linton (the notion is actually not intrinsic to $\mathcal{X}$, but rather depends on a specific presentation of $\mathcal{X}$). As in Lawvere’s case, these generalized theories participate in an explicit reflective adjunction with ∞ -categories over $\mathcal{X}$; that is, they afford a concrete answer to Question 2.0.1. The explicit constructions of derived ∞ -categories appearing later in the paper (for example the one described in Theorem 0.1.1) will come from this framework.

On the other hand, the notion of algebraic functor, being purely semantic in nature, generalizes

¹In this paper, “sifted” is generally to be read in the ∞ -categorical sense. However, this particular sentence is valid whether the word is read in the ∞ -categorical sense or in the 1-categorical sense, and of course the latter is closer to Lawvere’s original formulation.

immediately (as already seen in Definition 0.1.3). Moreover, experience suggests that it's a natural and useful notion in this greater generality. In §2.3, we'll discuss its relationship to the theories of §2.2. The practical complexity of this depends on $\mathcal{X}$ (and our chosen presentation of it), but for example, the case $\mathcal{X} = \mathbf{Spc}$ is very much analogous to Lawvere's (this is well known, coming down to the results of [27, §5.5.8]; we discuss it here in Remarks 2.3.13 and 2.3.15 and §5.1), and we'll also be able to analyze the case of a compactly generated stable ∞ -category in Ch. 4. This will allow us to prove algebraicity results for our constructions of derived ∞ -categories, as well as to characterize them in these terms (as in Theorem 0.1.4).

§2.1. $\mathcal{X}$ -objects. This section sets up some terminology that will expedite our discussion in the next one. We can frame it in this way: before contemplating algebraic structure on objects of a given presentable ∞ -category $\mathcal{X}$, we may consider $\mathcal{X}$ itself as articulating some type of mathematical structure; and we may ask: can we make sense of this structure internally to another ∞ -category $\mathcal{E}$? Definition 2.1.3 below is one way to do so which will be relevant here.

We'll see in Remark 2.1.7 that the notion just alluded to is intrinsic to $\mathcal{X}$ under the assumption that $\mathcal{E}$ is cocomplete. For our purposes, though, it is more practically formulated in terms of a specific presentation of $\mathcal{X}$; let us begin by stating what we mean by “presentation”.

Definition 2.1.1. By a *presented ∞ -category* we mean a presentable ∞ -category $\mathcal{X}$ together with a full subcategory $\mathcal{A} \subseteq \mathcal{X}$ that is essentially small and dense (Definition 1.2.5). We will often use the notation $\mathcal{X}|\mathcal{A}$ to refer such data; we recall from Example 1.2.10 that it determines a Bousfield localization

$$L_{\mathcal{X}|\mathcal{A}} : \mathcal{P}(\mathcal{A}) \rightleftarrows \mathcal{X} : j_{\mathcal{X}|\mathcal{A}},$$

which we think of as a presentation of $\mathcal{X}$.

Remark 2.1.2. Let $q : \mathcal{A} \rightarrow \mathcal{X}$ be a dense functor, and let $\mathcal{A}' \subseteq \mathcal{A}$ be a full subcategory such that every object of $\mathcal{A}$ is a retract of an object of $\mathcal{A}'$. Then the restriction $q|_{\mathcal{A}'} : \mathcal{A}' \rightarrow \mathcal{X}$ is also dense: this follows from the characterization of density in Proposition 1.2.9 and the fact that the restriction functor $\mathrm{Fun}(\mathcal{A}, \mathbf{Spc}) \rightarrow \mathrm{Fun}(\mathcal{A}', \mathbf{Spc})$ is an equivalence (as it admits a fully faithful left adjoint and is conservative).

For the remainder of the section, we fix a presentable ∞ -category $\mathcal{X}$ and a dense, full subcategory $\mathcal{A} \subseteq \mathcal{X}$. (In the subsequent sections, we'll primarily be interested in the situation of Definition 2.1.1, i.e. where $\mathcal{A}$ is also essentially small; but for getting the general sense of things in this preliminary discussion, it will be convenient to also consider the case $\mathcal{A} = \mathcal{X}$, so we don't yet make the smallness hypothesis.) We also fix a locally small ∞ -category $\mathcal{E}$.

Definition 2.1.3. An $\mathcal{X}|\mathcal{A}$ -object of $\mathcal{E}$ is a functor $\chi : \mathcal{A}^{\mathrm{op}} \rightarrow \mathcal{E}$ such that, for each $E \in \mathcal{E}$, the functor $\mathrm{Map}_{\mathcal{E}}(E, \chi(-)) : \mathcal{A}^{\mathrm{op}} \rightarrow \mathbf{Spc}$ is representable by an object of $\mathcal{X}$ (i.e. is in the essential image of $j_{\mathcal{X}|\mathcal{A}}$). We write $\mathcal{E}_{\mathcal{X}|\mathcal{A}} \subseteq \mathrm{Fun}(\mathcal{A}^{\mathrm{op}}, \mathcal{X})$ for the full subcategory spanned by the $\mathcal{X}|\mathcal{A}$ -objects. In the case that $\mathcal{A} = \mathcal{X}$, we will abbreviate $\mathcal{X}|\mathcal{A}$ to $\mathcal{X}$ in this terminology and notation.

Notation 2.1.4. Suppose given an $\mathcal{X}|\mathcal{A}$ -object $\chi : \mathcal{A}^{\mathrm{op}} \rightarrow \mathcal{E}$. For $E \in \mathcal{E}$, we denote by $\chi_E \in \mathcal{X}$ the object representing $\mathrm{Map}_{\mathcal{E}}(E, \chi(-))$, i.e. such that we have a natural equivalence $\mathrm{Map}_{\mathcal{E}}(E, \chi(A)) \simeq \mathrm{Map}_{\mathcal{X}}(A, \chi_E)$ for $A \in \mathcal{A}^{\mathrm{op}}$.

Remark 2.1.5. Definition 2.1.3 may be reformulated as follows. The restricted Yoneda functor $j_{\mathcal{X}|\mathcal{A}} : \mathcal{X} \rightarrow \mathrm{Fun}(\mathcal{A}^{\mathrm{op}}, \mathbf{Spc})$ induces a fully faithful functor

$$\mathrm{Fun}(\mathcal{E}^{\mathrm{op}}, \mathcal{X}) \hookrightarrow \mathrm{Fun}(\mathcal{E}^{\mathrm{op}}, \mathrm{Fun}(\mathcal{A}^{\mathrm{op}}, \mathbf{Spc})) \simeq \mathrm{Fun}(\mathcal{A}^{\mathrm{op}}, \mathrm{Fun}(\mathcal{E}^{\mathrm{op}}, \mathbf{Spc}))$$

and the Yoneda embedding $j_{\mathcal{E}} : \mathcal{E} \rightarrow \mathrm{Fun}(\mathcal{E}^{\mathrm{op}}, \mathbf{Spc})$ induces a fully faithful functor

$$\mathrm{Fun}(\mathcal{A}^{\mathrm{op}}, \mathcal{E}) \hookrightarrow \mathrm{Fun}(\mathcal{A}^{\mathrm{op}}, \mathrm{Fun}(\mathcal{E}^{\mathrm{op}}, \mathbf{Spc})).$$

The ∞ -category $\mathcal{E}_{\mathcal{X}|\mathcal{A}}$ fits into a cartesian diagram

$$\begin{array}{ccc} \mathcal{E}_{\mathcal{X}|\mathcal{A}} & \hookrightarrow & \mathrm{Fun}(\mathcal{A}^{\mathrm{op}}, \mathcal{E}) \\ \downarrow & & \downarrow \\ \mathrm{Fun}(\mathcal{E}^{\mathrm{op}}, \mathcal{X}) & \hookrightarrow & \mathrm{Fun}(\mathcal{A}^{\mathrm{op}}, \mathrm{Fun}(\mathcal{E}^{\mathrm{op}}, \mathrm{Spc})). \end{array}$$

Hence, an $\mathcal{X}|\mathcal{A}$ -object of $\mathcal{E}$ may equivalently be thought of as a functor $v : \mathcal{E}^{\mathrm{op}} \rightarrow \mathcal{X}$ such that, for each $A \in \mathcal{A}$, the functor $\mathrm{Map}_{\mathcal{X}}(A, v(-)) : \mathcal{E}^{\mathrm{op}} \rightarrow \mathrm{Spc}$ is representable (i.e. v admits a left adjoint at A).

Remark 2.1.6. An $\mathcal{X}|\mathcal{A}$ -object $\chi : \mathcal{A}^{\mathrm{op}} \rightarrow \mathcal{E}$ preserves any limit in $\mathcal{A}^{\mathrm{op}}$ whose corresponding colimit in $\mathcal{A}$ is preserved by the inclusion into $\mathcal{X}$. Conversely, if there is a collection $\mathfrak{C}$ of colimit diagrams in $\mathcal{A}$ such that the essential image of $j_{\mathcal{X}|\mathcal{A}} : \mathcal{X} \rightarrow \mathrm{Fun}(\mathcal{A}^{\mathrm{op}}, \mathrm{Spc})$ consists exactly of those presheaves that preserve the corresponding collection of limit diagrams in $\mathcal{A}^{\mathrm{op}}$, then a functor $\chi : \mathcal{A}^{\mathrm{op}} \rightarrow \mathcal{E}$ is an $\mathcal{X}|\mathcal{A}$ -object if and only if it preserves the same collection of limit diagrams.

Remark 2.1.7. Let $\mathfrak{K}$ be a class of small ∞ -categories, suppose that $\mathcal{E}$ admits $\mathfrak{K}$ -shaped colimits. Then, for any functor $v : \mathcal{E}^{\mathrm{op}} \rightarrow \mathcal{X}$, the collection of objects $X \in \mathcal{X}$ at which v admits a left adjoint is closed under $\mathfrak{K}$ -shaped colimits. Letting $\mathcal{A}'$ be the full subcategory of $\mathcal{X}$ generated under $\mathfrak{K}$ -shaped colimits by $\mathcal{A}$, it follows from Remark 2.1.5 that, $\mathcal{X}|\mathcal{A}$ -objects of $\mathcal{E}$ are equivalent to $\mathcal{X}|\mathcal{A}'$ -objects of $\mathcal{E}$; more precisely, the essential images of the embeddings of $\mathcal{E}_{\mathcal{X}|\mathcal{A}}$ and $\mathcal{E}_{\mathcal{X}|\mathcal{A}'}$ into $\mathrm{Fun}(\mathcal{E}^{\mathrm{op}}, \mathcal{X})$ are equal. Specializing to the case that $\mathfrak{K}$ contains all small ∞ -categories, we see that $\mathcal{E}_{\mathcal{X}|\mathcal{A}} \simeq \mathcal{E}_{\mathcal{X}}$ when $\mathcal{E}$ is cocomplete.

Example 2.1.8. Suppose $\mathcal{E} = \mathrm{Spc}$. This is cocomplete, so the choice of $\mathcal{A}$ does not matter by Remark 2.1.7. We have $\mathrm{Spc}_{\mathcal{X}} \simeq \mathcal{X}$: a functor $\chi : \mathcal{X}^{\mathrm{op}} \rightarrow \mathrm{Spc}$ is an $\mathcal{X}$ -object in Spc if and only if it is representable.

Example 2.1.9. Suppose $\mathcal{X} = \mathrm{Spc}$. Then we may take $\mathcal{A}$ to be the full subcategory $*$ consisting of just the one point space. For any locally small ∞ -category $\mathcal{E}$, we have $\mathcal{E}_{\mathrm{Spc}|*} \simeq \mathcal{E}$: $\mathrm{Spc}|*$ -objects of $\mathcal{E}$ are simply functors $\chi : * \rightarrow \mathcal{E}$, equivalently objects $E \in \mathcal{E}$.

In fact, we could take $\mathcal{A}$ to be any full subcategory that contains the one point space. For instance, we could also take it to be the full subcategory $\Sigma \subseteq \mathrm{Spc}$ spanned by the finite sets. A $\mathrm{Spc}|\Sigma$ -object of $\mathcal{E}$ is a finite product preserving functor $\chi : \Sigma^{\mathrm{op}} \rightarrow \mathcal{E}$, specifying which is equivalent to specifying in $\mathcal{E}$ an object E that has finite powers, i.e. such that the product $E^{\times n}$ in $\mathcal{E}$ exists for each $n \in \mathbb{Z}_{\geq 0}$ (this equivalence may be seen using Remark 2.1.5).

Example 2.1.10. Suppose $\mathcal{X} = \mathrm{Set}$. As in Example 2.1.9, we may take $\mathcal{A}$ to be any full subcategory containing the one element set. For $\mathcal{A} = *$ (resp. $\mathcal{A} = \Sigma$), specifying a $\mathrm{Set}|\mathcal{A}$ -object of $\mathcal{E}$ is equivalent to specifying a 0-truncated object $E \in \mathcal{E}$ (resp. a 0-truncated object that has finite powers in $\mathcal{E}$).

The alternative characterization of $\mathcal{X}|\mathcal{A}$ -objects in Remark 2.1.5 makes sense without the assumption that $\mathcal{A}$ is dense in $\mathcal{X}$, but there will be no need for that generality in this paper. On the other hand, there is a different generalization which will have some relevance below. Namely, Definition 2.1.3 makes sense for any dense functor to, rather than just a dense full subcategory of, $\mathcal{X}$. The reason that we did not formulate Definition 2.1.3 in this way is that this increase in generality is only apparent, in the sense of the following result.

Proposition 2.1.11. *Let $q : \mathcal{A}' \rightarrow \mathcal{X}$ be a dense functor. Then:*

1. *the essential image $q(\mathcal{A}') \subseteq \mathcal{X}$ is a dense subcategory;*
2. *a functor $\chi : q(\mathcal{A}')^{\mathrm{op}} \rightarrow \mathrm{Spc}$ is in the essential image of $j_{\mathcal{X}|q(\mathcal{A}')}$ if and only if its restriction $q^*(\chi) = \chi \circ q : (\mathcal{A}')^{\mathrm{op}} \rightarrow \mathrm{Spc}$ is in the essential image of $j_{\mathcal{X}|\mathcal{A}'}$.*

Proof. For 1 we refer to [28, Tag 03VZ]. Let us prove 2. The only if direction is immediate from the

commutativity of the diagram

$$\begin{array}{ccc}
 & \mathcal{X} & \\
 j_{\mathcal{X}|q(\mathcal{A}')} \swarrow & & \searrow j_{\mathcal{X}|\mathcal{A}'} \\
 \text{Fun}(q(\mathcal{A}')^{\text{op}}, \text{Spc}) & \xrightarrow{q^*} & \text{Fun}((\mathcal{A}')^{\text{op}}, \text{Spc}).
 \end{array}$$

For the converse, we assume that $q^*(\chi) \simeq j_{\mathcal{X}|\mathcal{A}'}(X)$ for some $X \in \mathcal{X}$. Setting $\bar{\chi} := j_{\mathcal{X}|q(\mathcal{A}')}(\bar{X})$, the above commutative triangle gives us an equivalence $q^*(\chi) \simeq q^*(\bar{\chi})$. Thus, letting q_* denote right Kan extension along q (which a priori produces large presheaves, which we accept), it suffices to prove that the unit maps $\eta : \chi \rightarrow q_* q^*(\chi)$ and $\bar{\eta} : \bar{\chi} \rightarrow q_* q^*(\bar{\chi})$ are both equivalences. The latter is a consequence of the following chain of natural equivalences for $A \in q(\mathcal{A}')$:

$$\begin{aligned}
 (q_* q^*(\bar{\chi}))(A) &\simeq \text{Map}_{\text{Fun}(q(\mathcal{A}')^{\text{op}}, \text{Spc})}(j_{q(\mathcal{A}')}(\bar{X}), q_* q^*(\bar{\chi})) \\
 &\simeq \text{Map}_{\text{Fun}(\mathcal{A}'^{\text{op}}, \text{Spc})}(q^*(j_{q(\mathcal{A}')}(\bar{X})), q^*(\bar{\chi})) \\
 &\simeq \text{Map}_{\text{Fun}(q(\mathcal{A}')^{\text{op}}, \text{Spc})}(j_{q(\mathcal{A}')}(\bar{X}), \bar{\chi}) \\
 &\simeq \bar{\chi}(A);
 \end{aligned}$$

the first and fourth equivalences here are from the Yoneda lemma, the second is from the adjunction between q^* and q_* , and the third is from the full faithfulness of the diagonal arrows in the commutative triangle above. This implies also that $q^*(\bar{\eta})$ is an equivalence, which by the triangle identity and the identification $q^*(\chi) \simeq q^*(\bar{\chi})$ then implies that $q^*(\eta)$ is an equivalence. Since $q : \mathcal{A}' \rightarrow q(\mathcal{A}')$ is essentially surjective by construction, q^* here is conservative, and so we deduce that η is also an equivalence, completing the argument. $\square$

§2.2. Theorization. Throughout this section, we fix a presented ∞ -category $\mathcal{X}|\mathcal{A}$ (Definition 2.1.1), giving us a Bousfield localization of presentable ∞ -categories

$$L_{\mathcal{X}|\mathcal{A}} : \mathcal{P}(\mathcal{A}) \rightleftarrows \mathcal{X} : j_{\mathcal{X}|\mathcal{A}}.$$

Our attention will be centered around a certain notion of $\mathcal{X}|\mathcal{A}$ -theory in this context. Intuitively, we regard this notion as a syntactic, or diagrammatic, formulation of algebraic structure relative to $\mathcal{X}$, specified in terms of operations with arities restricted by the choice of $\mathcal{A}$ (see Example 2.2.5 for the basis of this intuition). We will also prove, though, that this syntactic notion is faithfully captured by its semantics (Proposition 2.2.20).

Most of the material in this section has appeared already in previous literature. Theorem 2.2.31 may be an exception, and another caveat is our differing choices in notation and terminology. See Remark 2.2.6 for more detail.

Definition 2.2.1. We say that a functor $\chi : \mathcal{A}^{\text{op}} \rightarrow \mathcal{T}$ is an:

1. $\mathcal{A}$ -pretheory if it is essentially surjective;
2. $\mathcal{A}$ -theory if it is an $\mathcal{A}$ -pretheory and $\mathcal{T}$ is essentially small;
3. $\mathcal{X}|\mathcal{A}$ -theory if it is an $\mathcal{A}$ -theory and an $\mathcal{X}|\mathcal{A}$ -object (Definition 2.1.3).

Following Convention 0.3.9, we may sometimes refer to an $\mathcal{A}$ -pretheory just by its codomain $\mathcal{T}$, leaving the functor χ implicit. We denote by

$$\text{PThy}_{\mathcal{A}} \supseteq \text{Thy}_{\mathcal{A}} \supseteq \text{Thy}_{\mathcal{X}|\mathcal{A}}$$

the full subcategories of $\text{Cat}_{\mathcal{A}^{\text{op}}}$ spanned by the $\mathcal{A}$ -pretheories, $\mathcal{A}$ -theories, and $\mathcal{X}|\mathcal{A}$ -theories.

Remark 2.2.2. Let $\chi : \mathcal{A}^{\text{op}} \rightarrow \mathcal{T}$ be an $\mathcal{A}$ -pretheory. Then χ is an $\mathcal{A}$ -theory if and only if $\mathcal{T}$ is locally small (because $\mathcal{A}$ is essentially small and χ is essentially surjective).

Definition 2.2.3. Let $\chi : \mathcal{A}^{\text{op}} \rightarrow \mathcal{T}$ be an $\mathcal{A}$ -pretheory. A *model* of $\mathcal{T}$ in $\mathcal{X}$ is a functor $M : \mathcal{T} \rightarrow \text{Spc}$ such that $M \circ \chi : \mathcal{A}^{\text{op}} \rightarrow \text{Spc}$ is in the essential image of $j_{\mathcal{X}|\mathcal{A}}$. We let $\text{Mod}_{\mathcal{X}}(\mathcal{T})$ denote the full subcategory of

$\text{Fun}(\mathcal{T}, \text{Spc})$ spanned by the models of $\mathcal{T}$ in $\mathcal{X}$. There is then an essentially unique functor $u : \text{Mod}_{\mathcal{X}}(\mathcal{T}) \rightarrow \mathcal{X}$ making the square of ∞ -categories

$$\begin{array}{ccc} \text{Mod}_{\mathcal{X}}(\mathcal{T}) & \hookrightarrow & \text{Fun}(\mathcal{T}, \text{Spc}) \\ u \downarrow & & \downarrow \chi^* \\ \mathcal{X} & \xrightarrow{j_{\mathcal{X}|\mathcal{A}}} & \text{Fun}(\mathcal{A}^{\text{op}}, \text{Spc}) \end{array}$$

commute (the upper horizontal arrow being the inclusion functor), and furthermore this square is cartesian. We refer to the functor u as the *structure map*.

The construction $\mathcal{T} \mapsto \text{Mod}_{\mathcal{X}}(\mathcal{T})$ canonically extends to a functor

$$\text{Mod}_{\mathcal{X}} : \text{PThy}_{\mathcal{A}}^{\text{op}} \rightarrow \text{Cat}_{/\mathcal{X}},$$

namely the composition

$$\text{PThy}_{\mathcal{A}}^{\text{op}} \rightarrow (\text{Cat}_{\mathcal{A}^{\text{op}}/})^{\text{op}} \rightarrow \text{Cat}_{/\text{Fun}(\mathcal{A}^{\text{op}}, \text{Spc})} \rightarrow \text{Cat}_{/\mathcal{X}},$$

where the first arrow is the inclusion functor, the second is induced by the functor $\text{Fun}(-, \text{Spc}) : \text{Cat}^{\text{op}} \rightarrow \text{Cat}$, and the third is base change along $j_{\mathcal{X}|\mathcal{A}}$.

Remark 2.2.4. Definition 2.2.1.3 can be reformulated in terms of Definition 2.2.3 as follows: an $\mathcal{A}$ -theory $\chi : \mathcal{A}^{\text{op}} \rightarrow \mathcal{T}$ is an $\mathcal{X}|\mathcal{A}$ -theory if and only if the Yoneda embedding $j_{\mathcal{T}}^{\vee} : \mathcal{T}^{\text{op}} \rightarrow \text{Fun}(\mathcal{T}, \text{Spc})$ factors through $\text{Mod}_{\mathcal{X}}(\mathcal{T})$, i.e. for each $T \in \mathcal{T}$, the corepresented functor $\text{Map}_{\mathcal{T}}(T, -) : \mathcal{T} \rightarrow \text{Spc}$ is a model of $\mathcal{T}$ in $\mathcal{X}$.

Example 2.2.5. Definitions 2.2.1 and 2.2.3 generalize the definitions of Lawvere discussed in the introduction to this chapter. Namely, as in Example 2.1.10, we may take $\mathcal{X} = \text{Set}$ and $\mathcal{A} = \Sigma$. Then an $\mathcal{X}|\mathcal{A}$ -theory $\chi : \mathcal{A}^{\text{op}} \rightarrow \mathcal{T}$ is a classical algebraic theory and $\text{Mod}_{\mathcal{X}}(\mathcal{T})$ is the category of classical models thereof.

We could also take $\mathcal{X} = \text{Spc}$ and $\mathcal{A} = \Sigma$, as in Example 2.1.9. Then an $\mathcal{X}|\mathcal{A}$ -theory $\chi : \mathcal{A}^{\text{op}} \rightarrow \mathcal{T}$ is what we will refer to simply as an *algebraic theory*: the only difference between this and the notion of the previous paragraph is that the ∞ -category $\mathcal{T}$ here need not be a 1-category.

For elaborations on these examples, see Remark 2.3.13 and §5.1.

Remark 2.2.6. Analogues of Definitions 2.2.1 and 2.2.3 in the setting of classical category theory trace back to work of Linton [23] and have been studied more recently in work of Avery [4] and Bourke–Garner [9]. The present ∞ -categorical formulation is due to work of Henry–Meadows [18].

Beware that our terminology conflicts with these references. For instance, what is called an $\mathcal{A}$ -theory in [18] is equivalent to what we call an $\mathcal{X}|\mathcal{A}$ -theory here (not what we call an $\mathcal{A}$ -theory); our choice is intended to emphasize that the notion depends on the embedding of $\mathcal{A}$ in $\mathcal{X}$, which is helpful for example in distinguishing the two cases mentioned in Example 2.2.5. And what is called an $\mathcal{A}$ -pretheory in [18] is equivalent to what we call an $\mathcal{A}$ -theory (not what we call an $\mathcal{A}$ -pretheory); we allow large objects into the treatment here to facilitate the discussion below of the left adjoint to $\text{Mod}_{\mathcal{X}}$ (similar to [4]).

Let us make a few basic observations about the ∞ -categories of models defined above.

Remark 2.2.7. Let $\chi : \mathcal{A}^{\text{op}} \rightarrow \mathcal{T}$ be an $\mathcal{A}$ -theory. Then the functors $j_{\mathcal{X}|\mathcal{A}}$ and χ^* whose fiber product appears in Definition 2.2.3 are maps in Pr^{R} . Since the forgetful functor $\text{Pr}^{\text{R}} \rightarrow \text{Cat}$ creates limits, that fiber product may also be regarded as taking place in Pr^{R} . Passing to left adjoints, we get a pushout square in Pr^{L} :

$$\begin{array}{ccc} \text{Mod}_{\mathcal{X}}(\mathcal{T}) & \longleftarrow & \text{Fun}(\mathcal{T}, \text{Spc}) \\ \uparrow & & \uparrow \chi^! \\ \mathcal{X} & \xleftarrow{L_{\mathcal{X}|\mathcal{A}}} & \text{Fun}(\mathcal{A}^{\text{op}}, \text{Spc}). \end{array}$$

Proposition 2.2.8. *Let $\chi : \mathcal{A}^{\text{op}} \rightarrow \mathcal{T}$ be an $\mathcal{A}$ -pretheory, let $u : \text{Mod}_{\mathcal{X}}(\mathcal{T}) \rightarrow \mathcal{X}$ be the structure map, and let $j : \text{Mod}_{\mathcal{X}}(\mathcal{T}) \hookrightarrow \text{Fun}(\mathcal{T}, \text{Spc})$ be the inclusion functor. Then we have the following:*

1. Let $p : \mathcal{K} \rightarrow \text{Mod}_{\mathcal{X}}(\mathcal{T})$ be any diagram. Then j preserves the colimit of p if and only if $j_{\mathcal{X}|\mathcal{A}}$ preserves the colimit of $u \circ p$. Moreover, if these equivalent conditions hold, then u preserves the colimit of p .
2. If χ is an $\mathcal{X}|\mathcal{A}$ -theory, then the functors u and j are both left Kan extended along the Yoneda embedding $j_{\mathcal{T}}^{\vee} : \mathcal{T}^{\text{op}} \hookrightarrow \text{Mod}_{\mathcal{X}}(\mathcal{T})$ (see Remark 2.2.4).

Proof. Assertion 1 follows from the definitions and the fact that $\chi^* : \text{Fun}(\mathcal{T}, \text{Spc}) \rightarrow \text{Fun}(\mathcal{A}^{\text{op}}, \text{Spc})$ preserves colimits (see the cartesian square in Definition 2.2.3, and keep in mind also that $j_{\mathcal{X}|\mathcal{A}}$ and j are fully faithful). For 2, recall that the identity functor $\text{id} : \text{Fun}(\mathcal{T}, \text{Spc}) \rightarrow \text{Fun}(\mathcal{T}, \text{Spc})$ is left Kan extended from $\mathcal{T}^{\text{op}}$. This implies that the inclusion functor j is as well, and then by 1 it follows that u is too. $\square$

Corollary 2.2.9. Let $\chi : \mathcal{A}^{\text{op}} \rightarrow \mathcal{T}$ be an $\mathcal{X}|\mathcal{A}$ -theory. Let κ be a regular cardinal and suppose that each $A \in \mathcal{A}$ is κ -compact as an object of $\mathcal{X}$. Then:

1. $j_{\mathcal{T}}^{\vee}(T)$ is a κ -compact object of $\text{Mod}_{\mathcal{X}}(\mathcal{T})$ for each $T \in \mathcal{T}$;
2. u preserves κ -filtered colimits.

Proof. Immediate from Proposition 2.2.8.1. $\square$

Proposition 2.2.10. Let $\chi : \mathcal{A}^{\text{op}} \rightarrow \mathcal{T}$ be an $\mathcal{A}$ -theory. Then the structure map $u : \text{Mod}_{\mathcal{X}}(\mathcal{T}) \rightarrow \mathcal{X}$ is monadic.

Proof. The functor u admits a left adjoint by Remark 2.2.7; is conservative because it is a base change of $\chi^* : \text{Fun}(\mathcal{T}, \text{Spc}) \rightarrow \text{Fun}(\mathcal{A}^{\text{op}}, \text{Spc})$, which is conservative because χ is essentially surjective; and preserves the colimit of any u -split simplicial diagram by Proposition 2.2.8.1, since the colimit of a split simplicial diagram is preserved by any functor. Now we may apply the monadicity theorem to conclude. $\square$

We now move on to investigate Question 2.0.1, adopting the point of view that algebraic structure on objects of $\mathcal{X}$ is axiomatized by the notion of $\mathcal{X}|\mathcal{A}$ -theory. To begin, we observe that the functor $\text{Mod}_{\mathcal{X}} : \text{PThy}_{\mathcal{A}}^{\text{op}} \rightarrow \text{Cat}_{/\mathcal{X}}$ admits a left adjoint.

Construction 2.2.11. Let $\mathcal{Y} \in \text{Cat}_{/\mathcal{X}}$. Composing the structure map $u : \mathcal{Y} \rightarrow \mathcal{X}$ with the restricted Yoneda functor $j_{\mathcal{X}|\mathcal{A}} : \mathcal{X} \rightarrow \text{Fun}(\mathcal{A}^{\text{op}}, \text{Spc})$ defines by adjunction a functor $\bar{\chi} : \mathcal{A}^{\text{op}} \rightarrow \text{Fun}(\mathcal{Y}, \text{Spc})$, given by the formula

$$\bar{\chi}(A)(Y) = \text{Map}_{\mathcal{X}}(A, u(Y))$$

for $A \in \mathcal{A}^{\text{op}}$ and $Y \in \mathcal{Y}$. We define $\text{Str}_{\mathcal{A}}(\mathcal{Y}) \subseteq \text{Fun}(\mathcal{Y}, \text{Spc})$ to be the essential image of $\bar{\chi}$. Then $\bar{\chi}$ factors through an essential surjection $\chi : \mathcal{A}^{\text{op}} \rightarrow \text{Str}_{\mathcal{A}}(\mathcal{Y})$. We regard $\text{Str}_{\mathcal{A}}(\mathcal{Y})$ as an $\mathcal{A}$ -pretheory in this way, i.e. with structure map χ , and we refer to this as the *structural $\mathcal{A}$ -pretheory* associated to $\mathcal{Y}$ (or u).

Remark 2.2.12. In Construction 2.2.11, the functor $\bar{\chi} : \mathcal{A}^{\text{op}} \rightarrow \text{Fun}(\mathcal{Y}, \text{Spc})$ may also be described as the restriction of $j_u^{\vee} : \mathcal{X}^{\text{op}} \rightarrow \text{Fun}(\mathcal{Y}, \text{Spc})$. In addition, for an object $A \in \mathcal{A}$ at which u admits a left adjoint $f(A) \in \mathcal{Y}$, we have a canonical equivalence $\chi(A) \simeq j_{\mathcal{Y}}^{\vee}(f(A))$, which by the Yoneda lemma gives an equivalence

$$\text{Map}_{\text{Str}_{\mathcal{A}}(\mathcal{Y})}(\chi(A), \chi(A')) \simeq \chi(A')(f(A)) \simeq \text{Map}_{\mathcal{X}}(A', uf(A))$$

for any other $A' \in \mathcal{A}$.

Proposition 2.2.13. The construction $\mathcal{Y} \mapsto \text{Str}_{\mathcal{A}}(\mathcal{Y})$ canonically extends to a functor $\text{Str}_{\mathcal{A}} : \text{Cat}_{/\mathcal{X}} \rightarrow \text{PThy}_{\mathcal{A}}^{\text{op}}$ which is left adjoint to $\text{Mod}_{\mathcal{X}} : \text{PThy}_{\mathcal{A}}^{\text{op}} \rightarrow \text{Cat}_{/\mathcal{X}}$.

Proof. Recall from Definition 2.2.3 that $\text{Mod}_{\mathcal{X}}$ is the composition

$$\text{PThy}_{\mathcal{A}}^{\text{op}} \xrightarrow{a} (\text{Cat}_{\mathcal{A}^{\text{op}}/})^{\text{op}} \xrightarrow{b} \text{Cat}_{/\text{Fun}(\mathcal{A}^{\text{op}}, \text{Spc})} \xrightarrow{c} \text{Cat}_{/\mathcal{X}},$$

where a is the inclusion functor, b is induced by $\text{Fun}(-, \text{Spc}) : \text{Cat}^{\text{op}} \rightarrow \text{Cat}$, and c is base change along $j_{\mathcal{X}|\mathcal{A}} : \mathcal{X} \rightarrow \text{Fun}(\mathcal{A}^{\text{op}}, \text{Spc})$. Each of these functors admits a left adjoint:

- A left adjoint $a' : (\text{Cat}_{\mathcal{A}^{\text{op}}/})^{\text{op}} \rightarrow \text{PThy}_{\mathcal{A}}^{\text{op}}$ to a is opposite to a right adjoint of the inclusion functor $\text{PThy}_{\mathcal{A}} \hookrightarrow \text{Cat}_{\mathcal{A}^{\text{op}}/}$. Exhibiting the latter is equivalent to exhibiting a $\text{PThy}_{\mathcal{A}}$ -coreflection of any functor $\chi : \mathcal{A}^{\text{op}} \rightarrow \mathcal{T}$, which is provided by the factorization of χ through its essential image.

- Let us set $b_0 := \text{Fun}(-, \text{Spc}) : \text{Cat}^{\text{op}} \rightarrow \text{Cat}$ and denote the opposite functor by $b'_0 : \text{Cat} \rightarrow \text{Cat}^{\text{op}}$. The canonical natural equivalence

$$\text{Map}_{\text{Cat}}(\mathcal{T}, \text{Fun}(\mathcal{Y}, \text{Spc})) \simeq \text{Map}_{\text{Cat}}(\mathcal{Y}, \text{Fun}(\mathcal{T}, \text{Spc})),$$

for $\mathcal{T}, \mathcal{Y} \in \text{Cat}^{\text{op}}$ exhibits b'_0 as left adjoint to b_0 . This adjunction then induces an adjunction on slices $b' : \text{Cat}/\text{Fun}(\mathcal{A}^{\text{op}}, \text{Spc}) \rightleftarrows (\text{Cat}_{\mathcal{A}^{\text{op}}})^{\text{op}} : b$.

- The left adjoint to c is the functor $c' : \text{Cat}/\mathcal{X} \rightarrow \text{Cat}/\text{Fun}(\mathcal{A}^{\text{op}}, \text{Spc})$ given by composition with $j_{\mathcal{X}|\mathcal{A}}$.

It follows that the composition of left adjoints $a'b'c'$ defines a left adjoint to $\text{Mod}_{\mathcal{X}}$. This composition reproduces Construction 2.2.11 on objects, proving the claim. $\square$

Remark 2.2.14. Let us describe the unit and counit maps for the adjunction of Proposition 2.2.13 (which are determined by the proof above):

1. For $\mathcal{Y} \in \text{Cat}/\mathcal{X}$, the unit map

$$\eta : \mathcal{Y} \rightarrow \text{Mod}_{\mathcal{X}}(\text{Str}_{\mathcal{A}}(\mathcal{Y})) \subseteq \text{Fun}(\text{Str}_{\mathcal{A}}(\mathcal{Y}), \text{Spc})$$

in $\text{Cat}/\mathcal{X}$ has underlying functor given by the adjunct of the inclusion functor $\text{Str}_{\mathcal{A}}(\mathcal{Y}) \hookrightarrow \text{Fun}(\mathcal{Y}, \text{Spc})$, i.e. we have $\eta(Y)(T) \simeq T(Y)$ for $Y \in \mathcal{Y}$ and $T \in \text{Str}_{\mathcal{A}}(\mathcal{Y})$.

2. For an $\mathcal{A}$ -pretheory $\mathcal{T}$, the counit map in $\text{PThy}_{\mathcal{A}}^{\text{op}}$ corresponds to a map

$$\epsilon : \mathcal{T} \rightarrow \text{Str}_{\mathcal{A}}(\text{Mod}_{\mathcal{X}}(\mathcal{T})) \subseteq \text{Fun}(\text{Mod}_{\mathcal{X}}(\mathcal{T}), \text{Spc})$$

in $\text{PThy}_{\mathcal{A}}$. The underlying functor of ϵ is the adjunct of the inclusion functor $\text{Mod}_{\mathcal{X}}(\mathcal{T}) \hookrightarrow \text{Fun}(\mathcal{T}, \text{Spc})$, i.e. we have $\epsilon(T)(M) \simeq M(T)$ for $T \in \mathcal{T}$ and $M \in \text{Mod}_{\mathcal{X}}(\mathcal{T})$.

In the case that $\mathcal{T}$ is an $\mathcal{X}|\mathcal{A}$ -theory, which we recall is equivalent to the Yoneda embedding $j_{\mathcal{T}}^{\vee} : \mathcal{T}^{\text{op}} \rightarrow \text{Fun}(\mathcal{T}, \text{Spc})$ factoring through $\text{Mod}_{\mathcal{X}}(\mathcal{T})$ (Remark 2.2.4), the Yoneda lemma gives us a commutative diagram of ∞ -categories

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{\epsilon} & \text{Str}_{\mathcal{A}}(\text{Mod}_{\mathcal{X}}(\mathcal{T})) \\ j_{\mathcal{T}}^{\vee} \downarrow & & \downarrow \\ \text{Mod}_{\mathcal{X}}(\mathcal{T})^{\text{op}} & \xrightarrow{j_{\text{Mod}_{\mathcal{X}}(\mathcal{T})}^{\vee}} & \text{Fun}(\text{Mod}_{\mathcal{X}}(\mathcal{T}), \text{Spc}). \end{array}$$

This shows that the underlying functor of ϵ is fully faithful in this case; since the underlying functor of any map of $\mathcal{A}$ -pretheories is essentially surjective and the forgetful functor $\text{Cat}_{\mathcal{A}^{\text{op}}/} \rightarrow \text{Cat}$ is conservative, this implies that ϵ is an equivalence in this case.

Next we analyze how the subcategories $\text{Thy}_{\mathcal{X}|\mathcal{A}} \subseteq \text{Thy}_{\mathcal{A}} \subseteq \text{PThy}_{\mathcal{A}}$ fit into the above adjunction.

Definition 2.2.15. Let $\mathcal{Y} \in \text{Cat}/\mathcal{X}$. We say that $\mathcal{Y}$ is:

1. *$\mathcal{A}$ -theorizable* if $\text{Str}_{\mathcal{A}}(\mathcal{Y})$ is locally small, so that it is an $\mathcal{A}$ -theory (Remark 2.2.2);
2. *$\mathcal{A}$ -small* if $\mathcal{Y}$ is locally small and $\text{Str}_{\mathcal{A}}(\mathcal{Y})$ is contained in the full subcategory of copresheaves $\mathcal{P}^{\vee}(\mathcal{Y}) \subseteq \text{Fun}(\mathcal{Y}, \text{Spc})$ (Convention 0.3.10.b).

If $\mathcal{Y}$ is locally small, then so is $\mathcal{P}^{\vee}(\mathcal{Y})$, so $\mathcal{A}$ -smallness implies $\mathcal{A}$ -theorizability. We denote by

$$\text{Cat}_{/\mathcal{X}}^{\mathcal{A}\text{-thbl}} \supseteq \text{Cat}_{/\mathcal{X}}^{\mathcal{A}\text{-sm}}$$

the full subcategories of $\text{Cat}/\mathcal{X}$ spanned by the $\mathcal{A}$ -theorizable and $\mathcal{A}$ -small functors to ∞ -categories over $\mathcal{X}$.

Example 2.2.16. Let $\mathcal{Y}$ be a locally small ∞ -category and let $u : \mathcal{Y} \rightarrow \mathcal{X}$ be a functor admitting a partial left adjoint $f : \mathcal{A} \rightarrow \mathcal{Y}$ (for example, by Remark 2.2.7, we may take $\mathcal{Y} = \text{Mod}_{\mathcal{X}}(\mathcal{T})$ for $\mathcal{T}$ an $\mathcal{A}$ -theory and u to be the structure map). Then u is $\mathcal{A}$ -small, and hence $\mathcal{A}$ -theorizable, with $\text{Str}_{\mathcal{A}}(\mathcal{Y})$ identifying with $\mathcal{B}^{\text{op}}$, where $\mathcal{B}$ denotes the essential image of f . Under this identification, the structure map $\chi : \mathcal{A}^{\text{op}} \rightarrow \text{Str}_{\mathcal{A}}(\mathcal{Y})$ corresponds to the opposite of $f : \mathcal{A} \rightarrow \mathcal{B}$, and the unit map $\eta : \mathcal{Y} \rightarrow \text{Mod}_{\mathcal{X}}(\text{Str}_{\mathcal{A}}(\mathcal{Y}))$ (described in Remark 2.2.14.1) corresponds to the restricted Yoneda functor $j_{\mathcal{Y}|\mathcal{B}}$.

Example 2.2.17. Let $\mathcal{Y}$ be an accessible ∞ -category, so that $\mathcal{P}^\vee(\mathcal{Y}) \subseteq \text{Fun}(\mathcal{Y}, \text{Spc})$ is the full subcategory of accessible functors from $\mathcal{Y}$ to Spc . Then any accessible functor $u : \mathcal{Y} \rightarrow \mathcal{X}$ is $\mathcal{A}$ -small, and hence $\mathcal{A}$ -theorizable.

Proposition 2.2.18. *The adjunction of Proposition 2.2.13 restricts to an adjunction*

$$\text{Str}_{\mathcal{A}} : \text{Cat}_{/\mathcal{X}}^{\mathcal{A}\text{-thbl}} \rightleftarrows \text{Thy}_{\mathcal{A}}^{\text{op}} : \text{Mod}_{\mathcal{X}}.$$

Proof. The functor $\text{Str}_{\mathcal{A}} : \text{Cat}_{/\mathcal{X}} \rightarrow \text{PThy}_{\mathcal{A}}^{\text{op}}$ carries $\text{Cat}_{/\mathcal{X}}^{\mathcal{A}\text{-thbl}}$ into $\text{Thy}_{\mathcal{A}}^{\text{op}}$ by definition of $\mathcal{A}$ -theorizability, and the functor $\text{Mod}_{\mathcal{X}} : \text{PThy}_{\mathcal{A}}^{\text{op}} \rightarrow \text{Cat}_{/\mathcal{X}}$ carries $\text{Thy}_{\mathcal{A}}^{\text{op}}$ into $\text{Cat}_{/\mathcal{X}}^{\mathcal{A}\text{-thbl}}$ by Example 2.2.16. $\square$

The restricted adjunction of Proposition 2.2.18 is of a particularly nice kind; we recall the following general terminology to articulate this.

Definition 2.2.19. Let $l : \mathcal{U} \rightleftarrows \mathcal{V} : r$ be an adjunction of ∞ -categories. We say that $U \in \mathcal{U}$ (resp. $V \in \mathcal{V}$) is a *fixed point* of the adjunction if the unit map $\eta : U \rightarrow rl(U)$ (resp. the counit map $\epsilon : lr(V) \rightarrow V$) is an equivalence.

Let $\mathcal{U}' \subseteq \mathcal{U}$ and $\mathcal{V}' \subseteq \mathcal{V}$ be the full subcategories spanned by the fixed points of the adjunction. We say that the adjunction is *idempotent* if $l(U) \in \mathcal{V}'$ for all $U \in \mathcal{U}$, or equivalently $r(V) \in \mathcal{U}'$ for all $V \in \mathcal{V}$, or equivalently the adjunction factors as a reflective adjunction, adjoint equivalence, and coreflective adjunction as follows:

$$\mathcal{U} \xrightleftharpoons{rl} \mathcal{U}' \xrightleftharpoons[r]{l} \mathcal{V}' \xrightleftharpoons[lr]{} \mathcal{V}.$$

Proposition 2.2.20. *The adjunction of Proposition 2.2.18 is idempotent, and an $\mathcal{A}$ -theory $\mathcal{T}$ is a fixed point of the adjunction if and only if it is an $\mathcal{X}|\mathcal{A}$ -theory.*

Lemma 2.2.21. *Let $u : \mathcal{Y} \rightarrow \mathcal{X}$ be an $\mathcal{A}$ -theorizable functor. Then the $\mathcal{A}$ -theory $\text{Str}_{\mathcal{A}}(\mathcal{Y})$ is an $\mathcal{X}|\mathcal{A}$ -theory.*

Proof. Set $\mathcal{T} := \text{Str}_{\mathcal{A}}(\mathcal{Y})$ and let $\chi : \mathcal{A}^{\text{op}} \rightarrow \mathcal{T}$ be the structure map. We need to show that χ is an $\mathcal{X}|\mathcal{A}$ -object. That is, letting $T \in \mathcal{T}$, we need to show that the functor $\text{Map}_{\mathcal{T}}(T, \chi(-))$ is representable by an object of $\mathcal{X}$. Recall from Remark 2.2.12 that χ is a restriction of $j_u^\vee : \mathcal{X}^{\text{op}} \rightarrow \text{Fun}(\mathcal{Y}, \text{Spc})$; it therefore suffices to show that the functor $e := \text{Map}_{\text{Fun}(\mathcal{Y}, \text{Spc})}(T, j_u^\vee(-)) : \mathcal{X}^{\text{op}} \rightarrow \text{Spc}$ is representable.

Since $j_u^\vee$ preserves limits, so does e . The hypothesis that $\mathcal{Y}$ is $\mathcal{A}$ -theorizable, i.e. that $\mathcal{T}$ is locally small, gives us that $e(A)$ is essentially small for each $A \in \mathcal{A}^{\text{op}}$; since $\mathcal{A}^{\text{op}}$ generates $\mathcal{X}^{\text{op}}$ under limits, we deduce that e takes essentially small values. The previous two sentences, together with the presentability of $\mathcal{X}$, imply that e is representable. $\square$

Proof of Proposition 2.2.20. After Lemma 2.2.21, it suffices to show that any $\mathcal{X}|\mathcal{A}$ -theory $\mathcal{T}$ is a fixed point of the adjunction. This was established in Remark 2.2.14.2. $\square$

Proposition 2.2.20 tells us in particular that $\mathcal{X}|\mathcal{A}$ -theories are completely captured by their associated ∞ -categories of models in $\mathcal{X}$, more precisely that $\text{Mod}_{\mathcal{X}}$ identifies $\text{Thy}_{\mathcal{X}|\mathcal{A}}$ with a reflective subcategory of $\text{Cat}_{/\mathcal{X}}^{\mathcal{A}\text{-thbl}}$. To reinforce this perspective, we introduce notation and terminology for this subcategory and for its reflection.

Definition 2.2.22. Let $\mathcal{Y} \in \text{Cat}_{/\mathcal{X}}$. We say that $\mathcal{Y}$ is *$\mathcal{A}$ -theoretical* if in the essential image of $\text{Mod}_{\mathcal{X}} : \text{Thy}_{\mathcal{A}}^{\text{op}} \rightarrow \text{Cat}_{/\mathcal{X}}$, i.e. is equivalent to the ∞ -category of models of $\mathcal{T}$ in $\mathcal{X}$ for some $\mathcal{A}$ -theory $\mathcal{T}$. We denote by $\text{Cat}_{\mathcal{X}}^{\mathcal{A}\text{-th}} \subseteq \text{Cat}_{/\mathcal{X}}$ the full subcategory spanned by the $\mathcal{A}$ -theoretical ∞ -categories over $\mathcal{X}$.

Remark 2.2.23. By Proposition 2.2.20, the adjunction of Proposition 2.2.18 factors as a reflective adjunction, adjoint equivalence, and coreflective adjunction as follows:

$$\text{Cat}_{/\mathcal{X}}^{\mathcal{A}\text{-thbl}} \xrightleftharpoons{\text{Mod}_{\mathcal{X}} \text{Str}_{\mathcal{A}}} \text{Cat}_{/\mathcal{X}}^{\mathcal{A}\text{-th}} \xrightleftharpoons[\text{Mod}_{\mathcal{X}}]{\text{Str}_{\mathcal{A}}} \text{Thy}_{\mathcal{X}|\mathcal{A}}^{\text{op}} \xrightleftharpoons[\text{Str}_{\mathcal{A}} \text{Mod}_{\mathcal{X}}]{} \text{Thy}_{\mathcal{A}}^{\text{op}}.$$

Definition 2.2.24. Let $\mathcal{Y} \in \text{Cat}/\mathcal{X}$. By an $\mathcal{A}$ -theorization of $\mathcal{Y}$ (over $\mathcal{X}$), we mean a $\text{Cat}/\mathcal{X}^{\mathcal{A}\text{-th}}$ -reflection of it, i.e. an $\mathcal{A}$ -theoretical functor $\mathcal{Y}^{\mathcal{A}\text{-th}/\mathcal{X}} \rightarrow \mathcal{X}$ equipped with a functor $\eta : \mathcal{Y} \rightarrow \mathcal{Y}^{\mathcal{A}\text{-th}/\mathcal{X}}$ over $\mathcal{X}$ such that, for any $\mathcal{A}$ -theoretical functor $\mathcal{Z} \rightarrow \mathcal{X}$, precomposition with η defines an equivalence

$$\text{Map}_{\text{Cat}/\mathcal{X}}(\mathcal{Y}^{\mathcal{A}\text{-th}/\mathcal{X}}, \mathcal{Z}) \xrightarrow{\sim} \text{Map}_{\text{Cat}/\mathcal{X}}(\mathcal{Y}, \mathcal{Z}).$$

Remark 2.2.25. Let $\mathcal{Y} \in \text{Cat}/\mathcal{X}$. If $\mathcal{Y}$ is $\mathcal{A}$ -theorizable, then $\mathcal{Y}$ admits an $\mathcal{A}$ -theorization: we may take $\mathcal{Y}^{\mathcal{A}\text{-th}/\mathcal{X}} := \text{Mod}_{\mathcal{X}}(\text{Str}_{\mathcal{A}}(\mathcal{Y}))$ and $\eta : \mathcal{Y} \rightarrow \mathcal{Y}^{\mathcal{A}\text{-th}/\mathcal{X}}$ to be the unit map of the adjunction of Proposition 2.2.18 (see Remark 2.2.23). We will show in Proposition 3.2.15 that the converse holds as well (this is not really pertinent to the main narrative here, but it supports our terminology).

In the remainder of the section, we'll extract a few additional features of Definitions 2.2.22 and 2.2.24. We first formulate a criterion for equivalence between $\mathcal{A}$ -theoretical ∞ -categories over $\mathcal{X}$.

Proposition 2.2.26. *Suppose given a commutative triangle of ∞ -categories*

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{v} & \mathcal{Y}' \\ & \searrow u \quad \swarrow u' & \\ & \mathcal{X} & \end{array}$$

in which u and u' are both $\mathcal{A}$ -theoretical. Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ and $f' : \mathcal{X} \rightarrow \mathcal{Y}'$ be the left adjoints of u and u' , and let $t := uf$ and $t' := u'f'$ be the associated monads on $\mathcal{X}$. Let $\alpha : f' \rightarrow vf$ be the adjunct of the unit map $\text{id}_{\mathcal{X}} \rightarrow uf \simeq u'vf$. Then the following conditions are equivalent:

1. v is an equivalence;
2. the map $\alpha : f'(A) \rightarrow vf(A)$ is an equivalence for each $A \in \mathcal{A}$;
3. the map $u'(\alpha) : t'(A) = u'f'(A) \rightarrow u'vf(A) \simeq uf(A) = t(A)$ is an equivalence for each $A \in \mathcal{A}$.

Proof. The implications $1 \Rightarrow 2 \Rightarrow 3$ are clear. Let's prove that $3 \Rightarrow 1$. Let $\mathcal{T} := \text{Str}_{\mathcal{A}}(\mathcal{Y})$ and $\mathcal{T}' := \text{Str}_{\mathcal{A}}(\mathcal{Y}')$, and let $\chi : \mathcal{A}^{\text{op}} \rightarrow \mathcal{T}$ and $\chi' : \mathcal{A}^{\text{op}} \rightarrow \mathcal{T}'$ be the structure maps. Since u and u' are $\mathcal{A}$ -theoretical, v is an equivalence if and only if the induced map of $\mathcal{X}|\mathcal{A}$ -theories $\bar{v} := \text{Str}_{\mathcal{A}}(v) : \mathcal{T}' \rightarrow \mathcal{T}$ is an equivalence. Being a map of $\mathcal{X}|\mathcal{A}$ -theories, the functor $\bar{v}$ is necessarily essentially surjective, so the question is whether it is fully faithful, i.e. whether for each $A, B \in \mathcal{A}$ the map

$$\text{Map}_{\mathcal{T}'}(\chi'(A), \chi'(B)) \rightarrow \text{Map}_{\mathcal{T}}(\chi(A), \chi(B)).$$

induced by $\bar{v}$ is an equivalence. After unravelling definitions, one finds that this identifies with the map

$$\text{Map}_{\mathcal{X}}(A, t'(B)) \rightarrow \text{Map}_{\mathcal{X}}(A, t(B)),$$

induced by $u'(\alpha) : t' \rightarrow t$, hence is an equivalence if 3 holds. $\square$

Next we connect $\mathcal{A}$ -theorization with the concepts of §1.2.

Proposition 2.2.27. *Let $u : \mathcal{Y} \rightarrow \mathcal{X}$ be an $\mathcal{A}$ -theorizable functor, let $\mathcal{T} := \text{Str}_{\mathcal{A}}(\mathcal{Y}) \subseteq \text{Fun}(\mathcal{Y}, \text{Spc})$, let $\eta : \mathcal{Y} \rightarrow \mathcal{Y}^{\mathcal{A}\text{-th}/\mathcal{X}} = \text{Mod}_{\mathcal{X}}(\mathcal{T})$ be the $\mathcal{A}$ -theorization map, let $u' : \mathcal{Y}^{\mathcal{A}\text{-th}/\mathcal{X}} \rightarrow \mathcal{X}$ be the structure map, and let $j_{\mathcal{T}}^{\vee} : \mathcal{T}^{\text{op}} \rightarrow \mathcal{Y}^{\mathcal{A}\text{-th}/\mathcal{X}}$ be the Yoneda embedding. Then we have natural equivalences*

$$j_{\mathcal{T}}^{\vee} \simeq \Gamma_{\eta}, \quad u' \circ j_{\mathcal{T}}^{\vee} \simeq \Gamma_u$$

i.e. $j_{\mathcal{T}}^{\vee}$ is a partial right adjoint of the restricted Yoneda functor $j_{\eta}^{\vee} : \mathcal{Y}^{\mathcal{A}\text{-th}/\mathcal{X}} \rightarrow \text{Fun}(\mathcal{Y}, \text{Spc})^{\text{op}}$ and $u' \circ j_{\mathcal{T}}^{\vee}$ is a partial right adjoint of the restricted Yoneda functor $j_u^{\vee} : \mathcal{X} \rightarrow \text{Fun}(\mathcal{Y}, \text{Spc})^{\text{op}}$ (Notation 1.2.1).

Proof. The second equivalence follows from the first and the fact that u' admits a left adjoint f' (so that $j_u^{\vee} \simeq j_{\eta}^{\vee} \circ f'$). To establish the first equivalence, we begin by observing that, for $T \in \mathcal{T}$, we have a natural

equivalence $\alpha : T \simeq j_{\eta}^{\vee} j_{\mathcal{T}}^{\vee}(T)$ in $\text{Fun}(\mathcal{Y}, \text{Spc})$, i.e. that the diagram

$$(2.2.28) \quad \begin{array}{ccc} \mathcal{T}^{\text{op}} & \hookrightarrow & \text{Fun}(\mathcal{Y}, \text{Spc})^{\text{op}} \\ j_{\mathcal{T}}^{\vee} \downarrow & \nearrow j_{\eta}^{\vee} & \\ \mathcal{Y}^{\mathcal{A}\text{-th}/\mathcal{X}} & & \end{array}$$

commutes; this follows from the natural equivalences

$$\text{Map}_{\mathcal{Y}^{\mathcal{A}\text{-th}/\mathcal{X}}} (j_{\mathcal{T}}^{\vee}(T), \eta(Y)) \simeq \eta(Y)(T) \simeq T(Y)$$

for $T \in \mathcal{T}^{\text{op}}$ and $Y \in \mathcal{Y}$, the first of which is given by the Yoneda lemma, and the second by the definition of η (see Remark 2.2.14.1).

Let us now show that α exhibits $j_{\mathcal{T}}^{\vee}$ as a partial right adjoint of $j_{\eta}^{\vee}$, i.e. that the composition

$$\text{Map}_{\mathcal{Y}^{\mathcal{A}\text{-th}/\mathcal{X}}} (M, j_{\mathcal{T}}^{\vee}(T)) \xrightarrow{j_{\mathcal{T}}^{\vee}} \text{Map}_{\text{Fun}(\mathcal{Y}, \text{Spc})} (j_{\eta}^{\vee} j_{\mathcal{T}}^{\vee}(T), j_{\eta}^{\vee}(M)) \xrightarrow{\alpha} \text{Map}_{\text{Fun}(\mathcal{Y}, \text{Spc})} (T, j_{\eta}^{\vee}(M))$$

is an equivalence for $T \in \mathcal{T}^{\text{op}}$ and $M \in \mathcal{Y}^{\mathcal{A}\text{-th}/\mathcal{X}}$. Since $j_{\eta}^{\vee}$ preserves colimits and $\mathcal{Y}^{\mathcal{A}\text{-th}/\mathcal{X}} = \text{Mod}_{\mathcal{X}}(\mathcal{T})$ is generated under colimits by the essential image of $j_{\mathcal{T}}^{\vee}$, it suffices to show this when $M = j_{\mathcal{T}}^{\vee}(T')$ for $T' \in \mathcal{T}$. In this case, the above composition is inverse to the equivalence

$$\text{Map}_{\text{Fun}(\mathcal{Y}, \text{Spc})} (T, j_{\eta}^{\vee} j_{\mathcal{T}}^{\vee}(T')) \xrightarrow{\alpha} \text{Map}_{\mathcal{T}} (T, T') \xrightarrow{j_{\mathcal{T}}^{\vee}} \text{Map}_{\mathcal{Y}^{\mathcal{A}\text{-th}/\mathcal{X}}} (j_{\mathcal{T}}^{\vee}(T'), j_{\mathcal{T}}^{\vee}(T)),$$

completing the proof. $\square$

Remark 2.2.29. We record a couple of addenda to Proposition 2.2.27. In addition to the setup there, we let $\chi : \mathcal{A}^{\text{op}} \rightarrow \mathcal{T}$ be the structure map, we let $f' : \mathcal{X} \rightarrow \mathcal{Y}^{\mathcal{A}\text{-th}/\mathcal{X}}$ be the left adjoint of u' , and we observe that the commutative square of left adjoints in Remark 2.2.7 determines a natural equivalence $f'(A) \simeq j_{\mathcal{T}}^{\vee}(\chi(A))$ for $A \in \mathcal{A}$.

1. The proof above shows that the counit map $j_{\eta}^{\vee} \Gamma_{\eta}(T) \rightarrow T$ is (defined and) an equivalence for $T \in \mathcal{T}$. It follows that the unit map $M \rightarrow \Gamma_{\eta} j_{\eta}^{\vee}(M)$ is (defined and) an equivalence for M in the essential image of $j_{\mathcal{T}}^{\vee}$, or equivalently the essential image of $f'|_{\mathcal{A}}$.
2. The result gives us natural equivalences

$$u' f'(A) \simeq u' j_{\mathcal{T}}^{\vee}(\chi(A)) \simeq \Gamma_u(\chi(A)) = \Gamma_u(j_u^{\vee}(A))$$

for $A \in \mathcal{A}$. It follows from 1 that, under this identification, the two unit maps $A \rightarrow u' f'(A)$ and $A \rightarrow \Gamma_u j_u^{\vee}(A)$ are naturally equivalent.

Corollary 2.2.30. *Suppose given a commutative triangle of ∞ -categories*

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{v} & \mathcal{Z} \\ & \searrow u & \swarrow w \\ & \mathcal{X} & \end{array}$$

in which u is $\mathcal{A}$ -theorizable and w is $\mathcal{A}$ -theoretical. Let $h : \mathcal{X} \rightarrow \mathcal{Z}$ be the left adjoint of w and let $\mathcal{A}_{\mathcal{Z}} \subseteq \mathcal{Z}$ be the essential image of $\mathcal{A}$ under h . Then:

1. each object of $\mathcal{A}_{\mathcal{Z}}$ is v -coproximal (Definition 1.2.5);
2. the functor v exhibits $\mathcal{Z}$ as the $\mathcal{A}$ -theorization of $\mathcal{Y}$ if and only if each object of $\mathcal{A}_{\mathcal{Z}}$ is v -corealized, i.e. the unit map

$$h(A) \rightarrow \Gamma_v(j_v^{\vee}(h(A)))$$

is an equivalence for each $A \in \mathcal{A}$.

Proof. Set $\mathcal{T} := \text{Str}_{\mathcal{A}}(\mathcal{Y})$, let $\chi : \mathcal{A}^{\text{op}} \rightarrow \mathcal{T}$ be the structure map, and let $\eta : \mathcal{Y} \rightarrow \mathcal{Y}^{\mathcal{A}\text{-th}/\mathcal{X}} = \text{Mod}_{\mathcal{X}}(\mathcal{T})$ be the $\mathcal{A}$ -theorization map. Since w is $\mathcal{A}$ -theoretical, we have a unique functor $v' : \mathcal{Y}^{\mathcal{A}\text{-th}/\mathcal{X}} \rightarrow \mathcal{Z}$ over $\mathcal{X}$ with an equivalence $v \simeq v' \eta$. The functor v' creates limits (since its source and target are monadic over $\mathcal{X}$),

implying that $\Gamma_v \simeq v' \circ \Gamma_\eta$. In particular, since Γ_η is defined on $\mathcal{T}$ by Proposition 2.2.27, so is Γ_v . Noting that the adjunction between h and w gives a natural equivalence $j_v^\vee(h(A)) \simeq j_u^\vee(A) = \chi(A) \in \mathcal{T}$ for $A \in \mathcal{A}$, the previous sentence establishes 1.

We could prove 2 by applying Proposition 2.2.26, but let us instead phrase the argument more directly. The “only if” direction is given by Remark 2.2.29.1. Conversely, if the objects of $\mathcal{A}_Z$ are v -corealized, then the functor

$$j_v|_{\mathcal{A}_Z^{\text{op}}} : \mathcal{A}_Z^{\text{op}} \rightarrow \mathcal{T} \subseteq \text{Fun}(\mathcal{Y}, \text{Spc})$$

is fully faithful (Proposition 1.2.8). Under the identification $\mathcal{A}_Z^{\text{op}} \simeq \text{Str}_{\mathcal{A}}(Z)$ (Example 2.2.16), this functor recovers the map of $\mathcal{X}|\mathcal{A}$ -theories $\text{Str}_{\mathcal{A}}(v') : \text{Str}_{\mathcal{A}}(Z) \rightarrow \text{Str}_{\mathcal{A}}(\mathcal{Y})$. A fully faithful map of $\mathcal{X}|\mathcal{A}$ -theories is an equivalence, and since the source and target of v' are $\mathcal{A}$ -theoretical, this implies, by Proposition 2.2.20, that v' is an equivalence. $\square$

Finally, we have the following alternative characterization of $\mathcal{A}$ -theoretical ∞ -categories over $\mathcal{X}$, whose more practicable consequences will be discussed in the next section.

Theorem 2.2.31. *Let $\mathcal{Y}$ be a locally small ∞ -category and let $u : \mathcal{Y} \rightarrow \mathcal{X}$ be a functor. Then u is $\mathcal{A}$ -theoretical if and only if it satisfies the following three conditions:*

1. *It is conservative.*
2. *It admits a left adjoint $f : \mathcal{X} \rightarrow \mathcal{Y}$.*

For the third condition, recall from Example 2.2.16 that 2 implies that u is $\mathcal{A}$ -theorizable, with $\mathcal{A}$ -theorization $\mathcal{Y}^{\mathcal{A}\text{-th}/\mathcal{X}}$ given by $\text{Mod}_{\mathcal{X}}(\mathcal{T})$ for $\mathcal{T}$ the opposite of the essential image of $\mathcal{A}$ under f .

3. *Let $M \in \text{Mod}_{\mathcal{X}}(\mathcal{T})$ and let $p_M : \mathcal{T}_{/M}^{\text{op}} \rightarrow \mathcal{Y}$ be the composition*

$$\mathcal{T}_{/M}^{\text{op}} = \mathcal{T}^{\text{op}} \times_{\text{Mod}_{\mathcal{X}}(\mathcal{T})} \text{Mod}_{\mathcal{X}}(\mathcal{T})_{/M} \rightarrow \mathcal{T}^{\text{op}} \hookrightarrow \mathcal{Y},$$

where the first arrow is the projection functor and the second is the embedding of the essential image of $\mathcal{A}$ under f . Then p_M admits a colimit that is preserved by u .

Proof. That the first two conditions hold when u is $\mathcal{A}$ -theoretical is a part of Proposition 2.2.10. Thus, we may assume that these hold, and it remains to show that the unit map $\eta : \mathcal{Y} \rightarrow \mathcal{Y}^{\mathcal{A}\text{-th}/\mathcal{X}}$ is an equivalence if and only if 3 holds.

Recalling from Remark 2.2.14.1 that η identifies with the restricted Yoneda functor $j_{\mathcal{Y}|\mathcal{T}^{\text{op}}}$, the first condition in 3, that the diagrams p_M admit colimits, is equivalent to η admitting a left adjoint $\theta : \text{Mod}_{\mathcal{X}}(\mathcal{T}) \rightarrow \mathcal{Y}$, given by left Kan extension of the embedding $\mathcal{T}^{\text{op}} \hookrightarrow \mathcal{Y}$ along the Yoneda embedding $j_{\mathcal{T}}^\vee : \mathcal{T}^{\text{op}} \rightarrow \text{Mod}_{\mathcal{X}}(\mathcal{T})$. Assume that this holds, and let $\alpha : \text{id}_{\text{Mod}_{\mathcal{X}}(\mathcal{T})} \rightarrow \eta\theta$ be the unit transformation.

Let $u' : \text{Mod}_{\mathcal{X}}(\mathcal{T}) \rightarrow \mathcal{X}$ be the structure map, so that we have a commutative triangle

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{\eta} & \mathcal{Y}^{\mathcal{A}\text{-th}/\mathcal{X}} \simeq \text{Mod}_{\mathcal{X}}(\mathcal{T}) \\ & \searrow u & \downarrow u' \\ & & \mathcal{X}. \end{array}$$

Recalling from Proposition 2.2.8.2 that u' is left Kan extended from $\mathcal{T}^{\text{op}}$, the second condition in 3, that the colimits of the diagrams p_M are preserved by u , is equivalent to the map $u'(\alpha) : u' \rightarrow u'\eta\theta \simeq u\theta$ being an equivalence. Since u' is conservative, this is furthermore equivalent to α being an equivalence, i.e. the adjunction between η and θ being coreflective. But η is conservative, by the commutativity of the triangle above and the conservativity of u , so if the adjunction is coreflective, then in fact it is an equivalence. This proves that the entirety of condition 3 is equivalent to η being an equivalence. $\square$

§2.3. Algebraization. Let $\mathcal{X}$ be a presentable ∞ -category, fixed throughout this section. The axiomatization of algebraic structure discussed in the previous section has the advantage of being quite explicit: for any choice of essentially small, dense, full subcategory $\mathcal{A} \subseteq \mathcal{X}$, an $\mathcal{A}$ -theoretical ∞ -category over $\mathcal{X}$, i.e. the ∞ -category of models of an $\mathcal{X}|\mathcal{A}$ -theory, is defined by an explicit presentation/construction; and

consequently, approximating ∞ -categories over $\mathcal{X}$ by algebraic structure in this sense, i.e. $\mathcal{A}$ -theorization, is a transparent process.

We might simultaneously see a disadvantage here, though, in the extrinsic nature of these notions. This is related to their dependence on $\mathcal{A}$. In addition, though we have proved that $\mathcal{A}$ -theoreticity is a property of (rather than structure on) an ∞ -category over $\mathcal{X}$, it isn't immediate from the definitions how to detect this property in practice. Furthermore, some basic internal behavior of these ∞ -categories of models isn't obvious: for instance, how do we compute colimits in them?

We now consider a complementary approach to axiomatizing algebraic structure, which instead codifies some more obviously intrinsic properties of an ∞ -category over $\mathcal{X}$. Half of the following is a repetition of Definition 0.1.3.

Definition 2.3.1. We say that a functor $u : \mathcal{Y} \rightarrow \mathcal{X}$ is *algebraic* (resp. *strongly algebraic*) if it is conservative, admits a left adjoint, and creates sifted (resp. all) colimits. We denote by $\text{Cat}_{/\mathcal{X}}^{\text{alg}}$ (resp. $\text{Cat}_{/\mathcal{X}}^{\text{salg}}$) the full subcategory of $\text{Cat}_{/\mathcal{X}}$ spanned by the algebraic (resp. strongly algebraic) functors to ∞ -categories over $\mathcal{X}$.

Remark 2.3.2. Let $u : \mathcal{Y} \rightarrow \mathcal{X}$ be a functor. The monadicity theorem implies that, if u is algebraic, then it is monadic. Conversely, if u is monadic, then it is algebraic (resp. strongly algebraic) if and only if the associated monad preserves sifted (resp. all) colimits. It follows that, if u is algebraic, then $\mathcal{Y}$ is presentable (see e.g. [35, Proposition 4.1.10]).

Definition 2.3.3. Let $\mathcal{Y} \in \text{Cat}_{/\mathcal{X}}$. By an *algebraization* of $\mathcal{Y}$ (over $\mathcal{X}$), we mean a $\text{Cat}_{/\mathcal{X}}^{\text{alg}}$ -reflection of it, i.e. an algebraic functor $\mathcal{Y}^{\text{alg}/\mathcal{X}} \rightarrow \mathcal{X}$ equipped with a functor $\eta : \mathcal{Y} \rightarrow \mathcal{Y}^{\text{alg}/\mathcal{X}}$ over $\mathcal{X}$ such that, for any algebraic functor $\mathcal{Z} \rightarrow \mathcal{X}$, precomposition with η defines an equivalence

$$\text{Map}_{\text{Cat}_{/\mathcal{X}}}(\mathcal{Y}^{\text{alg}/\mathcal{X}}, \mathcal{Z}) \xrightarrow{\sim} \text{Map}_{\text{Cat}_{/\mathcal{X}}}(\mathcal{Y}, \mathcal{Z}).$$

Similarly, by a *strong algebraization* of $\mathcal{Y}$ we mean a $\text{Cat}_{/\mathcal{X}}^{\text{salg}}$ -reflection of it.

If we take Definition 2.3.1 as our codification of algebraic structure, then Question 2.0.1 may be reformulated in terms of Definition 2.3.3 as follows.

Question 2.3.4. *Given $\mathcal{Y} \in \text{Cat}_{/\mathcal{X}}$, does there exist an algebraization (or strong algebraization) of $\mathcal{Y}$ over $\mathcal{X}$? If so, can it be described explicitly?*

The work of Lawvere described in the introduction to this chapter says that a functor to Set is algebraic, in the sense of Definition 2.3.1, if and only if it is Σ -theoretical, in the sense of the previous subsection (see Example 2.2.5 and Definition 2.2.24). It follows that, for the case of algebraization, both parts of Question 2.3.4 have positive answers in the case $\mathcal{X} = \text{Set}$, under the mild hypothesis that $\mathcal{Y}$ is Σ -theorizable: then the algebraization is given by the Σ -theorization $\mathcal{Y}^{\Sigma\text{-th}/\text{Set}} = \text{Mod}_{\text{Set}}(\text{Str}_{\Sigma}(\mathcal{Y}))$. On the flip side, Lawvere's equivalence addresses the concerns about theoreticity enumerated at the beginning of this section (in the case $\mathcal{X} = \text{Set}$ and $\mathcal{A} = \Sigma$).

Our aim now is to understand how algebraicity and algebraization relate to theories and theorization for general $\mathcal{X}$.

Definition 2.3.5. We say that a full subcategory $\mathcal{A} \subseteq \mathcal{X}$ is Σ -dense if it is dense and closed under finite coproducts.

Notation 2.3.6. Let $\mathcal{A} \subseteq \mathcal{X}$ be an essentially small, Σ -dense, full subcategory. Then the Bousfield localization $L_{\mathcal{X}|\mathcal{A}} : \mathcal{P}(\mathcal{A}) \rightleftarrows \mathcal{X} : j_{\mathcal{X}|\mathcal{A}}$ factors as a composition of two Bousfield localizations

$$\mathcal{P}(\mathcal{A}) \xrightleftharpoons{L_{\Sigma}} \mathcal{P}_{\Sigma}(\mathcal{A}) \xrightleftharpoons[j_{\mathcal{X}|\mathcal{A}}]{L_{\mathcal{X}|\mathcal{A}}} \mathcal{X}.$$

(see Convention 0.3.10.d).

Remark 2.3.7. Let $\mathcal{A} \subseteq \mathcal{X}$ be an essentially small, dense, full subcategory and let $e : \mathcal{X} \rightarrow \mathcal{E}$ be a functor. If e preserves colimits, then e is left Kan extended from $\mathcal{A}$. If $\mathcal{A}$ is Σ -dense, then in fact it's sufficient for

e to preserve sifted colimits, because for any $X \in \mathcal{X}$, the ∞ -category $\mathcal{A}_{/X}$ admits finite coproducts and hence is sifted.

Remark 2.3.8. Let $\mathcal{A}_0 \subseteq \mathcal{X}$ be an essentially small, dense, full subcategory. Then the full subcategory $\mathcal{A} \subseteq \mathcal{X}$ generated under finite coproducts by $\mathcal{A}_0$ is essentially small and Σ -dense, and a presheaf $\chi : \mathcal{A}^{\text{op}} \rightarrow \mathbf{Spc}$ is in the essential image of $j_{\mathcal{X}|\mathcal{A}}$ if and only if it preserves finite products and its restriction to $\mathcal{A}_0$ is in the essential image of $j_{\mathcal{X}|\mathcal{A}_0}$. The essential smallness of $\mathcal{A}$ is clear, but let us justify the other claims.

Consider the full subcategory $\mathcal{A}' \subseteq \mathcal{P}(\mathcal{A}_0)$ generated under finite coproducts by the representable presheaves, and let $q : \mathcal{A}' \rightarrow \mathcal{X}$ be the restriction of $L_{\mathcal{X}|\mathcal{A}_0}$. Then q preserves finite coproducts, and we have a commutative diagram

$$\begin{array}{ccc} & \mathcal{X} & \\ j_{\mathcal{X}|\mathcal{A}'} \swarrow & & \searrow j_{\mathcal{X}|\mathcal{A}_0} \\ \mathcal{P}_{\Sigma}(\mathcal{A}') & \xrightarrow{q^*} & \mathcal{P}(\mathcal{A}_0). \end{array}$$

By [28, Tag 04BQ], the lower horizontal arrow here is an equivalence. Since the rightward diagonal arrow is fully faithful, we deduce that the leftward diagonal arrow is too, i.e. that q is dense and that the essential image of $j_{\mathcal{X}|\mathcal{A}'}$ consists of those finite product preserving presheaves on $\mathcal{A}'$ whose restriction to $\mathcal{A}_0$ is in the essential image of $j_{\mathcal{X}|\mathcal{A}_0}$. Our assertions about $\mathcal{A}$ now follow from Proposition 2.1.11 and the fact that it is the essential image of q .

Proposition 2.3.9. *Let $\mathcal{A} \subseteq \mathcal{X}$ be an essentially small, Σ -dense, full subcategory. Let $\mathcal{T}$ be a locally small ∞ -category and let $\chi : \mathcal{A}^{\text{op}} \rightarrow \mathcal{T}$ be an $\mathcal{X}|\mathcal{A}$ -object. Then:*

1. χ preserves finite products;

Moreover, if χ is also essentially surjective, hence an $\mathcal{X}|\mathcal{A}$ -theory, then:

2. $\mathcal{T}$ admits finite products, and a functor $e : \mathcal{T} \rightarrow \mathcal{E}$ preserves finite products if and only if $e\chi : \mathcal{A}^{\text{op}} \rightarrow \mathcal{E}$ does;
3. $\text{Mod}_{\mathcal{X}}(\mathcal{T}) \subseteq \mathcal{P}_{\Sigma}^{\vee}(\mathcal{T})$, i.e. every model of $\mathcal{T}$ in $\mathcal{X}$ preserves finite products.

Proof. Assertion 1 is a special case of Remark 2.1.6. Assertion 2 then follows from the fact that, if χ is essentially surjective, then any diagram $J \rightarrow \mathcal{T}$ where J is a finite set factors through χ . Finally, 3 is a consequence of the second clause in 2, the definition of models in $\mathcal{X}$, and the fact that the essential image of $j_{\mathcal{X}|\mathcal{A}} : \mathcal{X} \rightarrow \mathcal{P}(\mathcal{A})$ lies in $\mathcal{P}_{\Sigma}(\mathcal{A})$. $\square$

Theorem 2.3.10. *Let $\mathcal{A} \subseteq \mathcal{X}$ be an essentially small, Σ -dense (resp. dense), full subcategory. Then every algebraic (resp. strongly algebraic) ∞ -category over $\mathcal{X}$ is $\mathcal{A}$ -theoretical.*

Proof. We prove the Σ -dense/algebraic case; a similar argument applies to the other case. Let $u : \mathcal{Y} \rightarrow \mathcal{X}$ be an algebraic functor. By Remark 2.3.2, $\mathcal{Y}$ is presentable and hence locally small. We claim that u furthermore satisfies the three conditions of Theorem 2.2.31, and hence is $\mathcal{A}$ -theoretical. The first two conditions are part of the definition of algebraicity. The third condition follows from the rest of the definition of algebraicity, because the indexing ∞ -category $\mathcal{T}_{/M}^{\text{op}}$ of the diagram p_M there admits finite coproducts by Proposition 2.3.9, and hence is sifted. $\square$

Theorem 2.3.11. *Let $\mathcal{A} \subseteq \mathcal{X}$ be an essentially small, Σ -dense (resp. dense), full subcategory. Let $\chi : \mathcal{A}^{\text{op}} \rightarrow \mathcal{T}$ be an $\mathcal{X}|\mathcal{A}$ -theory, let $\mathcal{Y} := \text{Mod}_{\mathcal{X}}(\mathcal{T})$, let $u : \mathcal{Y} \rightarrow \mathcal{X}$ be the structure map, and let $t : \mathcal{X} \rightarrow \mathcal{X}$ be the associated monad (Proposition 2.2.10). Let $\mathfrak{J}$ be a class of essentially small ∞ -categories that contains the sifted ones (resp. the class of all essentially small ∞ -categories); let*

$$\text{Fun}'(\mathcal{Y}, \mathcal{X}) \subseteq \text{Fun}(\mathcal{Y}, \mathcal{X}), \quad \text{End}'(\mathcal{X}) \subseteq \text{End}(\mathcal{X})$$

be the full subcategories spanned by those functors that preserve $\mathfrak{J}$ -shaped colimits; and let

$$\text{Fun}'(\mathcal{T}^{\text{op}}, \mathcal{X}) \subseteq \text{Fun}(\mathcal{T}^{\text{op}}, \mathcal{X}), \quad \text{Fun}'(\mathcal{A}, \mathcal{X}) \subseteq \text{Fun}(\mathcal{A}, \mathcal{X})$$

be, respectively, the essential images of $\text{Fun}'(\mathcal{Y}, \mathcal{X})$ and $\text{End}'(\mathcal{X})$ under the functors $\text{Fun}(\mathcal{Y}, \mathcal{X}) \rightarrow \text{Fun}(\mathcal{T}^{\text{op}}, \mathcal{X})$

and $\text{End}(\mathcal{X}) \rightarrow \text{Fun}(\mathcal{A}, \mathcal{X})$ given by restriction along the Yoneda embedding $j_{\mathcal{T}}^{\vee} : \mathcal{T}^{\text{op}} \hookrightarrow \mathcal{Y}$ and the inclusion $i : \mathcal{A} \hookrightarrow \mathcal{X}$. Then the following conditions are equivalent:

1. $u \in \text{Fun}'(\mathcal{Y}, \mathcal{X})$;
2. $u|_{\mathcal{T}^{\text{op}}} \in \text{Fun}'(\mathcal{T}^{\text{op}}, \mathcal{X})$;
3. $t \in \text{End}'(\mathcal{X})$;
4. $t|_{\mathcal{A}} \in \text{Fun}'(\mathcal{A}, \mathcal{X})$.

Proof. The implications $1 \Rightarrow 3 \Rightarrow 4$ and $1 \Rightarrow 2 \Rightarrow 4$ are clear. We'll complete the proof by establishing the implication $4 \Rightarrow 1$.

Consider the coreflective adjunction

$$i_! : \text{Fun}(\mathcal{A}, \mathcal{X}) \rightleftarrows \text{End}(\mathcal{X}) : i^*$$

of left Kan extension and restriction along i . Let $\text{End}''(\mathcal{X}) \subseteq \text{End}(\mathcal{X})$ be the essential image of $i_!$. Let $t' := i_! i^*(t) \in \text{End}''(\mathcal{X})$, and let $\epsilon : t' \rightarrow t$ denote the counit map. We claim the following:

- a. $i^*(\epsilon)$ is an equivalence;
- b. $t' \in \text{End}'(\mathcal{X})$;
- c. t' canonically promotes to a monad on $\mathcal{X}$, and ϵ canonically promotes to a map of monads.

Assertion a is clear. Assertion b follows from our hypothesis 4, because, by Remark 2.3.7, every endofunctor of $\mathcal{X}$ that preserves $\mathfrak{J}$ -shaped colimits is in the essential image of $i_!$, and hence the adjunction above restricts to an equivalence $\text{Fun}'(\mathcal{A}, \mathcal{X}) \simeq \text{End}'(\mathcal{X})$.

For c, recall that a monad structure is by definition an algebra structure with respect to the composition monoidal structure on $\text{End}(\mathcal{X})$. The given such structure on t determines a monoidal structure on $\text{End}(\mathcal{X})_{/t}$, together with one on the forgetful functor $\text{End}(\mathcal{X})_{/t} \rightarrow \text{End}(\mathcal{X})$, inducing an equivalence $\text{Alg}(\text{End}(\mathcal{X})_{/t}) \simeq \text{Alg}(\text{End}(\mathcal{X}))_{/t}$ [26, §2.2.2]. With this in mind, our aim is to promote t' to an algebra object of $\text{End}(\mathcal{X})_{/t}$.

To do so, we first observe that ϵ exhibits t' as the final object of the slice ∞ -category $\text{End}''(\mathcal{X})_{/t} \subseteq \text{End}(\mathcal{X})_{/t}$. Then, using b and the fact that $\text{End}'(\mathcal{X}) \subseteq \text{End}''(\mathcal{X})$, we deduce that it is also a final object of $\text{End}'(\mathcal{X})_{/t} \subseteq \text{End}(\mathcal{X})_{/t}$. Since $\text{End}'(\mathcal{X})$ is closed under composition in $\text{End}(\mathcal{X})$ and contains the identity functor, the monoidal structure on $\text{End}(\mathcal{X})_{/t}$ restricts to a monoidal structure on $\text{End}'(\mathcal{X})_{/t}$. Now the claim follows from the fact that any final object in any monoidal ∞ -category $\mathcal{M}$ carries a canonical algebra structure (which is a consequence of the forgetful functor $\text{Alg}(\mathcal{M}) \rightarrow \mathcal{M}$ creating limits [26, Corollary 3.2.2.4]).

Let us now argue that 1 holds. Let $\mathcal{Y}'$ be the ∞ -category $\text{Mod}_{t'}(\mathcal{X})$ of modules over t' , with respect to the monad structure constructed above, and let $u' : \mathcal{Y}' \rightarrow \mathcal{X}$ be the structure map. We have an identification $\mathcal{Y} \simeq \text{Mod}_t(\mathcal{X})$ over $\mathcal{X}$ (as u is monadic), and so restriction along our map of monads $\epsilon : t' \rightarrow t$ defines a functor $v : \mathcal{Y} \rightarrow \mathcal{Y}'$ over $\mathcal{X}$. It follows from b that u' preserves $\mathfrak{J}$ -shaped colimits, so it suffices to prove that v is an equivalence. Since u' is $\mathcal{A}$ -theoretical by Theorem 2.3.10, we may apply Proposition 2.2.26 and a to conclude. $\square$

Remark 2.3.12. Let $\mathcal{A} \subseteq \mathcal{X}$ be an essentially small, Σ -dense (resp. dense), full subcategory, and let $\mathcal{Y}$ be a $\mathcal{A}$ -theorizable ∞ -category over $\mathcal{X}$. It follows from Theorem 2.3.10 that, if the $\mathcal{A}$ -theorization $\mathcal{Y}^{\mathcal{A}\text{-th}/\mathcal{X}}$ is algebraic (resp. strongly algebraic) over $\mathcal{X}$, then it is also an algebraization (resp. strong algebraization) of $\mathcal{Y}$ over $\mathcal{X}$. Theorem 2.3.11 isolates what needs to be checked for this to occur.

To finish here, let us spell out the simplest cases of the preceding results in the Σ -dense setting, including the original case of Lawvere.

Remark 2.3.13. Let $\mathcal{A} \subseteq \mathcal{X}$ be an essentially small, Σ -dense, full subcategory, and suppose that, for some $0 \leq n \leq \infty$, the embedding $j_{\mathcal{X}|\mathcal{A}} : \mathcal{X} \rightarrow \mathcal{P}_{\Sigma}(\mathcal{A})$ has essential image consisting of the n -truncated objects of $\mathcal{P}_{\Sigma}(\mathcal{A})$, i.e. identifies $\mathcal{X}$ with the ∞ -category of product preserving functors from $\mathcal{A}^{\text{op}}$ to the ∞ -category $\text{Spc}_{\leq n}$ of n -truncated spaces. Then the notion of an $\mathcal{X}|\mathcal{A}$ -object of a locally small ∞ -category $\mathcal{E}$ is particularly simple: it is a finite product preserving functor $\chi : \mathcal{A}^{\text{op}} \rightarrow \mathcal{E}$ such that $\chi(\mathcal{A})$ is an n -truncated

object of $\mathcal{E}$ for each $A \in \mathcal{A}$. It follows that the notion of an $\mathcal{X}|\mathcal{A}$ -theory is simple too: it is a locally small n -category $\mathcal{T}$ that admits finite products, together with an essentially surjective, finite product preserving functor $\chi : \mathcal{A}^{\text{op}} \rightarrow \mathcal{T}$.

Example 2.3.14. The basic example for which the hypothesis of Remark 2.3.13 holds is where $\mathcal{X} = \text{Spc}_{\leq n}^I$ and $\mathcal{A} = \Sigma^I$, for some set I : that $j_{\text{Spc}^I|\Sigma^I} : \text{Spc}^I \rightarrow \mathcal{P}_{\Sigma}(\Sigma^I)$ is an equivalence follows from [28, Tag 04BQ], since Σ^I is the full subcategory generated under finite coproducts by the representable presheaves in $\text{Spc}^I = \mathcal{P}(I)$ (repeating a point from Remark 2.3.8). When I consists of a single element and when $n = 0$ or $n = \infty$, we recover the cases noted in Example 2.2.5.

Combining this basic example with Theorem 2.3.10 supplies a characterization of all examples: see Proposition 2.3.17 below.

Remark 2.3.15. In the situation of Theorem 2.3.11, suppose that $\mathfrak{K}$ is the class of sifted small ∞ -categories and that the hypothesis of Remark 2.3.13 holds. Then $\text{Fun}'(\mathcal{A}, \mathcal{X}) = \text{Fun}(\mathcal{A}, \mathcal{X})$, so that condition 4 holds tautologically. Thus, the result says that every $\mathcal{A}$ -theoretical ∞ -category over $\mathcal{X}$ is algebraic, giving a full converse to Theorem 2.3.10.

This special case of Theorem 2.3.11 can also be proved by a more direct and elementary argument. When $\mathcal{X} = \text{Set}$ and $\mathcal{A} = \Sigma$, it recovers the result of Lawvere outlined at the beginning of the chapter.

Definition 2.3.16. Let $0 \leq n \leq \infty$ and let $\mathcal{Y}$ be a locally small n -category that admits sifted colimits. We say that an object $Y \in \mathcal{Y}$ is *compact n -projective* (or just *compact projective* in the case $n = \infty$) if its corepresented functor $\text{Map}_{\mathcal{Y}}(Y, -) : \mathcal{Y} \rightarrow \text{Spc}_{\leq n}$ preserves sifted colimits.

Proposition 2.3.17. *Let $0 \leq n \leq \infty$, let $\mathcal{Y}$ be a cocomplete, locally small n -category, and let $\mathcal{B} \subseteq \mathcal{Y}$ be an essentially small, full subcategory. Then the following conditions are equivalent:*

1. $\mathcal{B}$ is a detecting subcategory (Convention 0.3.11) of compact n -projective objects;
2. $\mathcal{B}$ is Σ -dense in $\mathcal{Y}$ and the essential image of $j_{\mathcal{Y}|\mathcal{B}} : \mathcal{Y} \rightarrow \mathcal{P}_{\Sigma}(\mathcal{B})$ consists of the n -truncated objects of $\mathcal{P}_{\Sigma}(\mathcal{B})$.

Proof. Choose a set $\{B_i\}_{i \in I}$ of objects of $\mathcal{B}$ that generates $\mathcal{B}$ under finite coproducts, and let $u : \mathcal{Y} \rightarrow \text{Spc}_{\leq n}^I$ be the functor corepresented by this set of objects. Then u admits a left adjoint $f : \text{Spc}_{\leq n}^I \rightarrow \mathcal{Y}$, namely the left Kan extension of the functor $I \rightarrow \mathcal{Y}$ given by $i \mapsto B_i$. Condition 1 is equivalent to u also preserving sifted colimits and being conservative, i.e. u being algebraic. Condition 2 is equivalent to u being Σ^I -theoretical. Their equivalence thus follows from Remark 2.3.15. $\square$

Remark 2.3.18. Proposition 2.3.17 in the case $n = 0$ is essentially due to Lawvere [22], and in the case $n = \infty$ to Lurie [27, Proposition 5.5.8.25].

Remark 2.3.19. Suppose that we are in the situation of Proposition 2.3.17 and that the equivalent conditions there are satisfied. Then an object of $\mathcal{Y}$ is compact n -projective if and only if it is a retract of an object of $\mathcal{B}$. The “if” direction is obvious; let’s prove the “only if” direction. Let $Y \in \mathcal{Y}$ be a compact n -projective object. By Proposition 2.3.17.2, we may find an equivalence $Y \xrightarrow{\sim} \text{colim}_{j \in \mathcal{J}} B_j$ in $\mathcal{Y}$, where $\mathcal{J}$ is sifted and $B_j \in \mathcal{B}$ for each $j \in \mathcal{J}$. Since Y is compact n -projective, this equivalence must factor through the canonical map $B_{j_0} \rightarrow \text{colim}_{j \in \mathcal{J}} B_j$ for some $j_0 \in \mathcal{J}$, which implies that Y is a retract of B_{j_0} .

It follows that the full subcategory $\overline{\mathcal{B}} \subseteq \mathcal{Y}$ spanned by all compact n -projective objects is essentially small, and clearly it is also a detecting subcategory of compact n -projective objects.

Ch. 3. Functoriality of theorization

This chapter continues from §2.2, where we reviewed an analogue of algebraic theories in the setting of any presented ∞ -category $\mathcal{X}|\mathcal{A}$, along with an accompanying notion of theorization for ∞ -categories over $\mathcal{X}$. Here we'll catalog some further functoriality properties of these constructions. In §§3.1–3.3, we'll discuss how they behave when the base $\mathcal{X}|\mathcal{A}$ is allowed to vary: §3.1 and §3.2 essentially constitute a second pass at the main points of §2.2 in this more flexible setting (this being deferred from the first pass, at the cost of some redundancy, due to the potentially obfuscating technical and notational requirements), while §3.3 articulates some more special, complementary phenomena. Then, in §3.4, we'll discuss a general way to define functors out of theorizations, which, in the context of our applications to derived categories later in the paper, we may think of as providing a theory of derived functors.

§3.1. Variance. We begin by specifying the type of variance between presented ∞ -categories that we will study.

Construction 3.1.1. We Prd^L denote the subcategory of $\text{Fun}(\Delta^1, \text{Cat})$ defined as follows:

- An object of Prd^L is a functor $i : \mathcal{A} \rightarrow \mathcal{X}$ such that $\mathcal{X}$ is presentable and i is the inclusion of an essentially small, dense, full subcategory. In other words, an object is a presented ∞ -category in the sense of Definition 2.1.1; as stated there, we will also use the notation $\mathcal{X}|\mathcal{A}$ for such an object.
- A map in Prd^L from an object $i' : \mathcal{A}' \rightarrow \mathcal{X}'$ to another object $i : \mathcal{A} \rightarrow \mathcal{X}$ is a commutative square

$$\begin{array}{ccc} \mathcal{A}' & \xrightarrow{\lambda_\circ} & \mathcal{A} \\ i' \downarrow & & \downarrow i \\ \mathcal{X}' & \xrightarrow{\lambda} & \mathcal{X} \end{array}$$

such that $\lambda : \mathcal{X}' \rightarrow \mathcal{X}$ is a left adjoint. The space of such is canonically equivalent to the space of left adjoint functors $\lambda : \mathcal{X}' \rightarrow \mathcal{X}$ such that $\lambda(\mathcal{A}) \subseteq \mathcal{A}'$; below, we will always assume maps in Prd^L to be of this form, and we will denote them by the same symbol as the underlying functor of presentable ∞ -categories (e.g. $\lambda : \mathcal{X}'|\mathcal{A}' \rightarrow \mathcal{X}|\mathcal{A}$).

We set $\text{Prd}^R := (\text{Prd}^L)^{\text{op}}$.

Notation 3.1.2. We let $e_0, e_1 : \text{Fun}(\Delta^1, \text{Cat}) \rightarrow \text{Cat}$ denote the evaluation functors. By definition of Prd^L , these restrict to functors $e_0^L : \text{Prd}^L \rightarrow \text{Cat}^{\text{sm}}$ and $e_1^L : \text{Prd}^L \rightarrow \text{Pr}^L$, given on objects by $e_0^L(\mathcal{X}|\mathcal{A}) = \mathcal{A}$ and $e_1^L(\mathcal{X}|\mathcal{A}) = \mathcal{X}$. We let $e_0^R : \text{Prd}^R \rightarrow (\text{Cat}^{\text{sm}})^{\text{op}}$ denote the opposite of e_0^L , and we let $e_1^R : \text{Prd}^R \rightarrow \text{Pr}^R$ denote the composition

$$\text{Prd}^R = (\text{Prd}^L)^{\text{op}} \xrightarrow{e_1^L} (\text{Pr}^L)^{\text{op}} \simeq \text{Pr}^R.$$

We can now restate our notational convention for maps in Prd^L by saying that we identify them with their images under e_1^L . We will similarly identify maps in Prd^R with their images under e_1^R . That is, given a map $\lambda : \mathcal{X}|\mathcal{A} \rightarrow \mathcal{X}'|\mathcal{A}'$ in Prd^L , so that we have a functor of presentable ∞ -categories $\lambda : \mathcal{X}' \rightarrow \mathcal{X}$ which admits a right adjoint $\rho : \mathcal{X} \rightarrow \mathcal{X}'$ and which satisfies $\lambda(\mathcal{A}') \subseteq \mathcal{A}$, we denote the corresponding map in Prd^R by $\rho : \mathcal{X}|\mathcal{A} \rightarrow \mathcal{X}'|\mathcal{A}'$; beware that ρ does not necessarily carry $\mathcal{A}'$ into $\mathcal{A}$. We may also denote both adjoints simultaneously by $\lambda : \mathcal{X}'|\mathcal{A}' \rightleftarrows \mathcal{X}|\mathcal{A} : \rho$ and refer to this as a *presented adjunction*.

Remark 3.1.3. Let $\lambda : \mathcal{X}'|\mathcal{A}' \rightleftarrows \mathcal{X}|\mathcal{A} : \rho$ be a presented adjunction. We then have commutative squares

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{j_{\mathcal{X}|\mathcal{A}}} & \mathcal{P}(\mathcal{A}) \\ \rho \downarrow & & \downarrow \lambda^* \\ \mathcal{X}' & \xrightarrow{j_{\mathcal{X}'|\mathcal{A}'}} & \mathcal{P}(\mathcal{A}') \end{array} \quad \begin{array}{ccc} \mathcal{X} & \xleftarrow{L_{\mathcal{X}|\mathcal{A}}} & \mathcal{P}(\mathcal{A}) \\ \lambda \uparrow & & \uparrow \lambda_! \\ \mathcal{X}' & \xleftarrow{L_{\mathcal{X}'|\mathcal{A}'}} & \mathcal{P}(\mathcal{A}') \end{array}$$

(the two related by passing to adjoints). A natural condition to impose in this setting is that the left square be vertically left adjointable, or equivalently that the right square be horizontally right adjointable,

i.e. that the induced transformation $\lambda_! \circ j_{\mathcal{X}'|\mathcal{A}'} \rightarrow j_{\mathcal{X}|\mathcal{A}} \circ \lambda$ be an equivalence. This will not be needed for much of our discussion, and so we have not included it in the definition of the ∞ -categories Prd^L and Prd^R . But it will play a role in some of §3.3 (see Definition 3.3.7).

Let us record the naturality of the squares considered in Remark 3.1.3, which will be required in the next section.

Construction 3.1.4. The functor $e_0^L : \text{Prd}^L \rightarrow \text{Cat}^{\text{sm}}$ admits a left adjoint, carrying an essentially small ∞ -category $\mathcal{A}$ to its cocompletion $\mathcal{P}(\mathcal{A})|\mathcal{A} \in \text{Prd}^L$ (here we identify $\mathcal{A}$ with its essential image under the Yoneda embedding for convenience). We denote this left adjoint by $\mathcal{P}^L : \text{Cat}^{\text{sm}} \rightarrow \text{Prd}^L$. We denote the counit map for this adjunction by $L : \mathcal{P}^L \circ e_0^L \rightarrow \text{id}_{\text{Prd}^L}$; it is given at an object $\mathcal{X}|\mathcal{A} \in \text{Prd}^L$ by the map $L_{\mathcal{X}|\mathcal{A}} : \mathcal{P}(\mathcal{A})|\mathcal{A} \rightarrow \mathcal{X}|\mathcal{A}$.

We write $\mathcal{P}^R : (\text{Cat}^{\text{sm}})^{\text{op}} \rightarrow \text{Prd}^R$ for the opposite of $\mathcal{P}^L$, which is right adjoint to the opposite e_0^R of e_0^L . We write $j : \text{id}_{\text{Prd}^R} \rightarrow \mathcal{P}^R \circ e_0^R$ for the unit map of this adjunction, which is given at an object $\mathcal{X}|\mathcal{A} \in \text{Prd}^R$ by the map $j_{\mathcal{X}|\mathcal{A}} : \mathcal{X}|\mathcal{A} \rightarrow \mathcal{P}(\mathcal{A})|\mathcal{A}$.

Remark 3.1.5. The composition $e_1^R \circ \mathcal{P}^R : (\text{Cat}^{\text{sm}})^{\text{op}} \rightarrow \text{Pr}^R$ canonically identifies with the restriction of $\text{Fun}(-, \text{Spc}) : \text{Cat}^{\text{op}} \rightarrow \text{Cat}$; see [36, Corollary 2.5].

We now assemble $\mathcal{A}$ -pretheories, $\mathcal{A}$ -theories, and $\mathcal{X}|\mathcal{A}$ -theories, for varying presented ∞ -categories $\mathcal{X}|\mathcal{A}$, into ∞ -categories over Prd^L .

Construction 3.1.6. We let $\bar{e}_0^L : \text{Prd}^L \rightarrow \text{Cat}$ denote the composition of e_0^L with $(-)^{\text{op}} : \text{Cat} \rightarrow \text{Cat}$, and we let $\text{Cat}_{\bar{e}_0^L//}$ denote the pullback of the cospan of ∞ -categories

$$\text{Prd}^L \xrightarrow{\bar{e}_0^L} \text{Cat} \xleftarrow{e_0} \text{Fun}(\Delta^1, \text{Cat}).$$

An object of $\text{Cat}_{\bar{e}_0^L//}$ is given by a pair $(\mathcal{X}|\mathcal{A}, \mathcal{T})$ where $\mathcal{X}|\mathcal{A} \in \text{Prd}^L$ and $\mathcal{T} \in \text{Cat}_{\mathcal{A}^{\text{op}}/}$. For two such objects $(\mathcal{X}|\mathcal{A}, \mathcal{T})$ and $(\mathcal{X}'|\mathcal{A}', \mathcal{T}')$, a map $\phi : (\mathcal{X}'|\mathcal{A}', \mathcal{T}') \rightarrow (\mathcal{X}|\mathcal{A}, \mathcal{T})$ in $\text{Cat}_{\bar{e}_0^L//}$ consists of a map $\lambda : \mathcal{X}'|\mathcal{A}' \rightarrow \mathcal{X}|\mathcal{A}$ in Prd^L and a functor $\mu : \mathcal{T}' \rightarrow \mathcal{T}$ making the square

$$\begin{array}{ccc} (\mathcal{A}')^{\text{op}} & \xrightarrow{\lambda} & \mathcal{A}^{\text{op}} \\ \chi' \downarrow & & \downarrow \chi \\ \mathcal{T}' & \xrightarrow{\mu} & \mathcal{T}. \end{array}$$

commute (where χ' and χ are the structure maps).

We let $\text{PThy} \supseteq \text{pThy} \supseteq \text{Thy}$ denote the full subcategories of $\text{Cat}_{\bar{e}_0^L//}$ spanned by those objects $(\mathcal{X}|\mathcal{A}, \mathcal{T})$ such that $\mathcal{T}$ is an $\mathcal{A}$ -pretheory, $\mathcal{A}$ -theory, and $\mathcal{X}|\mathcal{A}$ -theory, respectively. Thus, we have projection functors from these ∞ -categories to Prd^L whose fibers over $\mathcal{X}|\mathcal{A} \in \text{Prd}^L$ are the ∞ -categories $\text{PThy}_{\mathcal{A}} \supseteq \text{Thy}_{\mathcal{A}} \supseteq \text{Thy}_{\mathcal{X}|\mathcal{A}}$ of $\mathcal{A}$ -pretheories, $\mathcal{A}$ -theories, and $\mathcal{X}|\mathcal{A}$ -theories, respectively. Taking opposites gives ∞ -categories over Prd^R with the opposite fibers.

The remainder of the section concerns cartesianness properties of the ∞ -categories over Prd^L just constructed.

Construction 3.1.7. Let $\lambda : \mathcal{X}'|\mathcal{A}' \rightleftarrows \mathcal{X}|\mathcal{A} : \rho$ be a presented adjunction and let $\chi : \mathcal{A}^{\text{op}} \rightarrow \mathcal{T}$ be an $\mathcal{A}$ -pretheory. Let $\mathcal{T}' \subseteq \mathcal{T}$ be the essential image of $(\mathcal{A}')^{\text{op}}$ under $\chi \circ \lambda^{\text{op}}$, and let $\chi' : (\mathcal{A}')^{\text{op}} \rightarrow \mathcal{T}'$ be the factorization of $\chi \circ \lambda^{\text{op}}$ through its essential image, so that χ' is an $\mathcal{A}'$ -pretheory. We refer to χ' (or $\mathcal{T}'$) as the *restriction along λ* , or alternatively the *left pushforward along ρ* , of χ (or $\mathcal{T}$). By definition, we have a commutative diagram

$$\begin{array}{ccc} (\mathcal{A}')^{\text{op}} & \xrightarrow{\lambda^{\text{op}}} & \mathcal{A}^{\text{op}} \\ \chi' \downarrow & & \downarrow \chi \\ \mathcal{T}' & \hookrightarrow & \mathcal{T}, \end{array}$$

which defines a map $\phi_{\lambda, \mathcal{T}} : (\mathcal{X}'|\mathcal{A}', \mathcal{T}') \rightarrow (\mathcal{X}|\mathcal{A}, \mathcal{T})$ in PThy lying over the map λ in Prd^{L} .

Proposition 3.1.8. *Let $p : \text{PThy} \rightarrow \text{Prd}^{\text{L}}$ be the projection functor. Then, in the situation of Construction 3.1.7, we have the following:*

1. *The map $\phi_{\lambda, \mathcal{T}}$ is a p -cartesian lift of λ .*
2. *If $\mathcal{T}$ is an $\mathcal{A}$ -theory (resp. $\mathcal{X}|\mathcal{A}$ -theory), then $\mathcal{T}'$ is an $\mathcal{A}'$ -theory (resp. $\mathcal{X}'|\mathcal{A}'$ -theory).*

Proof. Claim 1 is straightforward from the construction. For claim 2, we observe first that $\mathcal{T}'$ inherits local smallness from $\mathcal{T}$. Second, assuming that χ is $\mathcal{X}|\mathcal{A}$ -object, then for any $T \in \mathcal{T}'$ we have natural equivalences

$$\text{Map}_{\mathcal{T}'}(T, \chi'(A)) \simeq \text{Map}_{\mathcal{T}}(T, \chi\lambda(A)) \simeq \text{Map}_{\mathcal{X}}(\lambda(A), \chi_T) \simeq \text{Map}_{\mathcal{X}'}(A, \rho(\chi_T))$$

for $A \in (\mathcal{A}')^{\text{op}}$, demonstrating that χ' is an $\mathcal{X}'|\mathcal{A}'$ -object (with $\chi'_T \simeq \rho(\chi_T)$). $\square$

Corollary 3.1.9. *The projection functors from PThy , pThy , and Thy to Prd^{L} are cartesian fibrations, and the inclusion functors $\text{Thy} \hookrightarrow \text{pThy} \hookrightarrow \text{PThy}$ preserve cartesian maps.*

Notation 3.1.10. In the situation of Construction 3.1.7, Proposition 3.1.8.1 extends restriction along λ /left pushforward along ρ to a functor; we denote it by $\lambda^* = \rho_! : \text{PThy}_{\mathcal{A}} \rightarrow \text{PThy}_{\mathcal{A}'}$. Proposition 3.1.8.2 implies that this restricts to a functor $\lambda^* = \rho_! : \text{Thy}_{\mathcal{X}|\mathcal{A}} \rightarrow \text{Thy}_{\mathcal{X}'|\mathcal{A}'}$.

Construction 3.1.11. Let $\lambda : \mathcal{X}'|\mathcal{A}' \rightleftarrows \mathcal{X}|\mathcal{A} : \rho$ be a presented adjunction and let $\chi' : (\mathcal{A}')^{\text{op}} \rightarrow \mathcal{T}'$ be an $\mathcal{A}'$ -pretheory. We may form a pushout diagram

$$\begin{array}{ccc} (\mathcal{A}')^{\text{op}} & \xrightarrow{\lambda^{\text{op}}} & \mathcal{A}^{\text{op}} \\ \chi' \downarrow & & \downarrow \chi \\ \mathcal{T}' & \xrightarrow{\mu} & \mathcal{T} \end{array}$$

in Cat . Then χ is essentially surjective, i.e. an $\mathcal{A}$ -pretheory: this follows from the universal property of the pushout and the essential surjectivity of χ' , which together imply that the identity map of $\mathcal{T}$ factors through the essential image of χ . We refer to χ (or $\mathcal{T}$) as the *left pushforward along λ* of χ' (or $\mathcal{T}'$), and we write $\phi'_{\lambda, \mathcal{T}'} : (\mathcal{X}'|\mathcal{A}', \mathcal{T}') \rightarrow (\mathcal{X}|\mathcal{A}, \mathcal{T})$ for the map in PThy (over the map λ in Prd^{L}) defined by the above pushout square.

Proposition 3.1.12. *Let $p : \text{PThy} \rightarrow \text{Prd}^{\text{L}}$ be the projection functor. Then, in the situation of Construction 3.1.11, we have the following:*

1. *The map $\phi'_{\lambda, \mathcal{T}'}$ is a p -cocartesian lift of λ .*
2. *If $\mathcal{T}'$ is an $\mathcal{A}'$ -theory, then $\mathcal{T}$ is an $\mathcal{A}$ -theory.*

Proof. Claim 1 follows from the universal property of the pushout. Claim 2 follows from the fact that Cat^{sm} is closed under pushouts in Cat . $\square$

Corollary 3.1.13. *The projection functors from PThy and pThy to Prd^{L} are cocartesian fibrations, and the inclusion functor $\text{pThy} \hookrightarrow \text{PThy}$ preserves cocartesian maps.*

Notation 3.1.14. In the situation of Construction 3.1.11, Proposition 3.1.12.1 extends left pushforward along λ to a functor, which we denote by $\lambda_! : \text{PThy}_{\mathcal{A}'} \rightarrow \text{PThy}_{\mathcal{A}}$, and which is left adjoint to the functor λ^* of Notation 3.1.10. Proposition 3.1.12.2 implies that $\lambda_!$ carries $\mathcal{A}'$ -theories to $\mathcal{A}$ -theories, so that we have a restricted adjunction $\lambda_! : \text{Thy}_{\mathcal{A}'} \rightleftarrows \text{Thy}_{\mathcal{A}} : \lambda^*$.

Remark 3.1.15. By Remark 2.2.23, for each presented ∞ -category $\mathcal{X}|\mathcal{A}$, the inclusion $\text{Thy}_{\mathcal{X}|\mathcal{A}} \hookrightarrow \text{Thy}_{\mathcal{A}}$ admits a left adjoint/reflection. Thus, the adjunction $\lambda_! : \text{Thy}_{\mathcal{A}'} \rightleftarrows \text{Thy}_{\mathcal{A}} : \lambda^*$ of Notation 3.1.14 induces an adjunction $\lambda_{!!} : \text{Thy}_{\mathcal{X}'|\mathcal{A}'} \rightleftarrows \text{Thy}_{\mathcal{X}|\mathcal{A}} : \lambda^*$, where $\lambda_{!!}$ is the essentially unique such functor making the

diagram

$$\begin{array}{ccc} \mathrm{Thy}_{\mathcal{A}'} & \xrightarrow{\lambda_!} & \mathrm{Thy}_{\mathcal{A}} \\ \downarrow & & \downarrow \\ \mathrm{Thy}_{\mathcal{X}'|\mathcal{A}'} & \xrightarrow{\lambda_{!!}} & \mathrm{Thy}_{\mathcal{X}|\mathcal{A}} \end{array}$$

(in which the vertical arrows are the reflections) commute.

Given Corollary 3.1.9, the previous paragraph implies that the projection functor $\mathrm{Thy} \rightarrow \mathrm{Prd}^{\mathrm{L}}$ is also a cocartesian fibration, and that the reflections above are the fibers of a relative left adjoint over $\mathrm{Prd}^{\mathrm{L}}$ of the inclusion $\mathrm{Thy} \hookrightarrow \mathrm{pThy}$, with the relative left adjoint preserving cocartesian maps (see [26, Proposition 7.3.2.6]).

§3.2. Contravariance. For a presented ∞ -category $\mathcal{X}|\mathcal{A}$ and an $\mathcal{A}$ -pretheory $\mathcal{T}$, we have defined an ∞ -category $\mathrm{Mod}_{\mathcal{X}}(\mathcal{T})$ of models of $\mathcal{T}$ in $\mathcal{X}$ (Definition 2.2.3). The contravariant functoriality of this construction extends to the setting of varying $\mathcal{X}|\mathcal{A}$ formulated in §3.1 as follows.

Construction 3.2.1. We let $\mathrm{Cat}_{//e_1^{\mathrm{R}}}$ denote the pullback of the cospan of ∞ -categories

$$\mathrm{Prd}^{\mathrm{R}} \xrightarrow{e_1^{\mathrm{R}}} \mathrm{Cat} \xleftarrow{e_1} \mathrm{Fun}(\Delta^1, \mathrm{Cat}).$$

An object of $\mathrm{Cat}_{//e_1^{\mathrm{R}}}$ is given by a pair $(\mathcal{X}|\mathcal{A}, \mathcal{Y})$ where $\mathcal{X}|\mathcal{A} \in \mathrm{Prd}^{\mathrm{R}}$ and $\mathcal{Y} \in \mathrm{Cat}_{/\mathcal{X}}$. For two such objects $(\mathcal{X}|\mathcal{A}, \mathcal{Y})$ and $(\mathcal{X}'|\mathcal{A}', \mathcal{Y}')$, a map $\psi : (\mathcal{X}|\mathcal{A}, \mathcal{Y}) \rightarrow (\mathcal{X}'|\mathcal{A}', \mathcal{Y}')$ in $\mathrm{Cat}_{//e_1^{\mathrm{R}}}$ consists of a map $\rho : \mathcal{X}|\mathcal{A} \rightarrow \mathcal{X}'|\mathcal{A}'$ in $\mathrm{Prd}^{\mathrm{R}}$ and a functor $\sigma : \mathcal{Y} \rightarrow \mathcal{Y}'$ making the square

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{\sigma} & \mathcal{Y}' \\ u \downarrow & & \downarrow u' \\ \mathcal{X} & \xrightarrow{\rho} & \mathcal{X}' \end{array}$$

commute (where u and u' are the structure maps).

Remark 3.2.2. The functor $e_1 : \mathrm{Fun}(\Delta^1, \mathrm{Cat}) \rightarrow \mathrm{Cat}$ is a bicartesian fibration. Firstly, by [28, Tag 01UU and Tag 05PK], it is a cocartesian fibration and the covariant transport functor associated to a map $\alpha : \mathcal{X}_0 \rightarrow \mathcal{X}_1$ in Cat identifies with the functor of slice ∞ -categories

$$\alpha_! : \mathrm{Cat}_{/\mathcal{X}_0} \rightarrow \mathrm{Cat}_{/\mathcal{X}_1}$$

given by composition with α . Secondly, each such functor $\alpha_!$ admits a right adjoint

$$\alpha^* : \mathrm{Cat}_{/\mathcal{X}_1} \rightarrow \mathrm{Cat}_{/\mathcal{X}_0},$$

given by base change along α , from which it follows that e_1 is also a cartesian fibration, with contravariant transport functors given by these right adjoints [28, Tag 02FP].

By construction, we have a projection functor $\mathrm{Cat}_{//e_1^{\mathrm{R}}} \rightarrow \mathrm{Prd}^{\mathrm{R}}$ which is a base change of e_1 , and hence is also a bicartesian fibration, with covariant and contravariant transport functors as described in the previous paragraph.

Construction 3.2.3. We will construct a functor

$$\mathrm{Mod} : \mathrm{PThy}^{\mathrm{op}} \rightarrow \mathrm{Cat}_{//e_1^{\mathrm{R}}}$$

over $\mathrm{Prd}^{\mathrm{R}}$, which on fibers over $\mathcal{X}|\mathcal{A} \in \mathrm{Prd}^{\mathrm{R}}$ recovers the functor $\mathrm{Mod}_{\mathcal{X}} : \mathrm{PThy}_{\mathcal{A}}^{\mathrm{op}} \rightarrow \mathrm{Cat}_{/\mathcal{X}}$ of Definition 2.2.3.

We begin with the commutative diagram

$$\begin{array}{ccccc}
\text{Cat}_{\mathcal{E}_0^L}^{\text{op}} & \xrightarrow{q} & \text{Fun}(\Delta^1, \text{Cat})^{\text{op}} & \xrightarrow{\text{Fun}(-, \text{Spc})} & \text{Fun}(\Delta^1, \text{Cat}) \\
p \downarrow & & e_0 \downarrow & & \downarrow e_1 \\
\text{Prd}^R & \xrightarrow{e_0^R} & (\text{Cat}^{\text{sm}})^{\text{op}} & \xrightarrow{\text{Fun}(-, \text{Spc})} & \text{Cat} \\
& & \mathcal{P}^R \downarrow & & \downarrow \text{id} \\
& & \text{Prd}^R & \xrightarrow{e_1^R} & \text{Cat}.
\end{array}$$

Here, p and q are the projection functors, so that the upper left square is opposite to the cartesian square defining $\text{Cat}_{\mathcal{E}_0^L}$; the upper right square is clear; and the lower square is as described in Remark 3.1.5. The boundary of this diagram defines a functor $s : \text{Cat}_{\mathcal{E}_0^L} \rightarrow \text{Cat}_{\mathcal{E}_1^R}$ whose composition with the projection functor $r : \text{Cat}_{\mathcal{E}_1^R} \rightarrow \text{Prd}^R$ identifies with $\mathcal{P}^R \circ e_0^R \circ p$; on objects, the functor s is given by $(\mathcal{X}|\mathcal{A}, \mathcal{T}) \mapsto (\mathcal{P}(\mathcal{A})|\mathcal{A}, \text{Fun}(\mathcal{T}, \text{Spc}))$. Using that r is a cartesian fibration (Remark 3.2.2), the functor s contravariantly transports along the natural transformation $j : \text{id}_{\text{Prd}^R} \rightarrow \mathcal{P}^R \circ e_0^R$ of Construction 3.1.4 to a functor $t : \text{Cat}_{\mathcal{E}_0^L} \rightarrow \text{Cat}_{\mathcal{E}_1^R}$ over Prd^R . Restricting t to PThy^{op} gives the desired functor Mod .

Remark 3.2.4. Let us spell out the behavior of the functor $\text{Mod} : \text{PThy}^{\text{op}} \rightarrow \text{Cat}_{\mathcal{E}_1^R}$ of Construction 3.2.3 on maps (we will invoke the descriptions of maps in the source and target ∞ -categories from Constructions 3.1.6 and 3.2.1, respectively). Let $\phi : (\mathcal{X}'|\mathcal{A}', \mathcal{T}') \rightarrow (\mathcal{X}|\mathcal{A}, \mathcal{T})$ be a map in PThy , consisting of a map $\lambda : \mathcal{X}'|\mathcal{A}' \rightarrow \mathcal{X}|\mathcal{A}$ in Prd^L and a functor $\mu : \mathcal{T}' \rightarrow \mathcal{T}$ making the following square commute:

$$\begin{array}{ccc}
(\mathcal{A}')^{\text{op}} & \xrightarrow{\lambda} & \mathcal{A}^{\text{op}} \\
\chi' \downarrow & & \downarrow \chi \\
\mathcal{T}' & \xrightarrow{\mu} & \mathcal{T};
\end{array}$$

here χ and χ' are the structure maps. Then we have the following commutative diagram in Pr^R :

$$\begin{array}{ccccc}
& & \text{Fun}(\mathcal{T}, \text{Spc}) & \xrightarrow{\mu^*} & \text{Fun}(\mathcal{T}', \text{Spc}) \\
& \nearrow & \downarrow \chi^* & \nearrow & \downarrow (\chi')^* \\
\text{Mod}_{\mathcal{X}}(\mathcal{T}) & \xrightarrow{\mu^*} & \text{Mod}_{\mathcal{X}'}(\mathcal{T}') & & \\
\downarrow v & & \downarrow v' & & \\
& \text{Fun}(\mathcal{A}^{\text{op}}, \text{Spc}) & \xrightarrow{\lambda^*} & \text{Fun}((\mathcal{A}')^{\text{op}}, \text{Spc}). \\
& \nearrow j_{\mathcal{X}|\mathcal{A}} & \nearrow j_{\mathcal{X}'|\mathcal{A}'} & & \\
\mathcal{X} & \xrightarrow{\rho} & \mathcal{X}' & &
\end{array}$$

In the front square here, v and v' denote the structure maps, ρ is the right adjoint of λ , and the upper horizontal arrow comes from the observation that μ^* carries models of $\mathcal{T}$ in $\mathcal{X}$ to models of $\mathcal{T}'$ in $\mathcal{X}'$, which is a consequence of the definition of model (i.e. the cartesianness of the left and right squares) and the commutativity of the rest of the diagram. This front square determines a map $\psi : (\mathcal{X}|\mathcal{A}, \text{Mod}_{\mathcal{X}}(\mathcal{T})) \rightarrow (\mathcal{X}'|\mathcal{A}', \text{Mod}_{\mathcal{X}'}(\mathcal{T}'))$ in $\text{Cat}_{\mathcal{E}_1^R}$, lying over the map $\rho : \mathcal{X}|\mathcal{A} \rightarrow \mathcal{X}'|\mathcal{A}'$ in Prd^R ; this map ψ is equivalent to the image of ϕ (or rather the map in PThy^{op} corresponding to ϕ) under Mod .

Next we consider the construction $\mathcal{Y} \mapsto \text{Str}_{\mathcal{A}}(\mathcal{Y})$ (Construction 2.2.11), associating an $\mathcal{A}$ -pretheory to an ∞ -category over $\mathcal{X}$, for any $\mathcal{X}|\mathcal{A} \in \text{Prd}^R$. We established in Proposition 2.2.13 that this defines a left adjoint to the functor $\text{Mod}_{\mathcal{X}} : \text{PThy}_{\mathcal{A}}^{\text{op}} \rightarrow \text{Cat}_{/\mathcal{X}}$. In fact, these adjunctions for varying $\mathcal{X}|\mathcal{A}$ fit together into an adjunction over Prd^R :

Proposition 3.2.5. *The functor $\text{Mod} : \text{PThy}^{\text{op}} \rightarrow \text{Cat}_{\mathcal{E}_1^R}$ admits a left adjoint relative to Prd^R (which on fibers over $\mathcal{X}|\mathcal{A} \in \text{Prd}^R$ recovers the functor $\text{Str}_{\mathcal{A}} : \text{Cat}_{/\mathcal{X}} \rightarrow \text{PThy}_{\mathcal{A}}^{\text{op}}$ of Proposition 2.2.13).*

Proof. Since the projection functors $\text{PThy}^{\text{op}} \rightarrow \text{Prd}^R$ and $\text{Cat}_{\mathcal{E}_1^R} \rightarrow \text{Prd}^R$ are cocartesian fibrations (Corollary 3.1.9 and Remark 3.2.2), it suffices to verify the criterion of [26, Proposition 7.3.2.11]; let us spell it out in the present situation.

Let $\rho : \mathcal{X}|\mathcal{A} \rightarrow \mathcal{X}'|\mathcal{A}'$ be a map in Prd^R and let $\lambda : \mathcal{X}'|\mathcal{A}' \rightarrow \mathcal{X}|\mathcal{A}$ be the corresponding map in Prd^L . Let $\mathcal{Y} \in \text{Cat}_{/\mathcal{X}}$. Let $\mathcal{T} := \text{Str}_{\mathcal{A}}(\mathcal{Y})$ and let $\mathcal{T}'$ be the restriction along λ /left pushforward along ρ of $\mathcal{T}$, so that we have a canonical map $\phi : (\mathcal{X}'|\mathcal{A}', \mathcal{T}') \rightarrow (\mathcal{X}|\mathcal{A}, \mathcal{T})$ in PThy lying over λ (Construction 3.1.7), corresponding to a map in PThy^{op} lying over ρ . Then we have maps

$$\mathcal{Y} \xrightarrow{\eta} \text{Mod}_{\mathcal{X}}(\mathcal{T}) \xrightarrow{\text{Mod}(\phi)} \text{Mod}_{\mathcal{X}'}(\mathcal{T}')$$

in $\text{Cat}_{//e_1^R}$, where η is the unit map of the adjunction $\text{Str}_{\mathcal{A}} : \text{Cat}_{/\mathcal{X}} \rightleftarrows \text{PThy}_{\mathcal{A}}^{\text{op}} : \text{Mod}_{\mathcal{X}}$. The composition lies over the map ρ in Prd^R and hence defines a map $\rho_!(\mathcal{Y}) \rightarrow \text{Mod}_{\mathcal{X}'}$ in $\text{Cat}_{/\mathcal{X}'}$, which then determines by adjunction a map of $\mathcal{A}'$ -pretheories $\phi' : \mathcal{T}' \rightarrow \mathcal{T}''$, where $\mathcal{T}'' := \text{Str}_{\mathcal{A}'}(\rho_!(\mathcal{Y}))$. What we need to verify is that ϕ' is an equivalence.

Letting $u : \mathcal{Y} \rightarrow \mathcal{X}$ denote the structure map, the above claim follows from unwinding the definitions and invoking the natural equivalence

$$\text{Map}_{\mathcal{X}}(\lambda(A), u(Y)) \simeq \text{Map}_{\mathcal{X}'}(A, \rho u(Y))$$

for $A \in (\mathcal{A}')^{\text{op}}$ and $Y \in \mathcal{Y}$ determined by the adjunction between λ and ρ . $\square$

Notation 3.2.6. We let $\text{Str} : \text{Cat}_{//e_1^R} \rightarrow \text{PThy}^{\text{op}}$ denote the left adjoint over Prd^R of Proposition 3.2.5.

Notation 3.2.7. We let $\text{Cat}_{//e_1^R}^{\text{th}} \subseteq \text{Cat}_{//e_1^R}^{\text{thbl}}$ denote the full subcategories of $\text{Cat}_{//e_1^R}$ spanned by those objects $(\mathcal{X}|\mathcal{A}, \mathcal{Y})$ such that $\mathcal{Y}$ is $\mathcal{A}$ -theorizable or $\mathcal{A}$ -theoretical, respectively.

Proposition 3.2.8. *The adjunction of Proposition 3.2.5 restricts to an idempotent adjunction*

$$\text{Str} : \text{Cat}_{//e_1^R}^{\text{thbl}} \rightleftarrows \text{PThy}^{\text{op}} : \text{Mod},$$

which factors as a reflective adjunction, adjoint equivalence, and coreflective adjunction as follows:

$$\text{Cat}_{//e_1^R}^{\text{thbl}} \xrightleftharpoons{\text{Mod} \circ \text{Str}} \text{Cat}_{//e_1^R}^{\text{th}} \xrightleftharpoons[\text{Mod}]{\text{Str}} \text{Thy}^{\text{op}} \xrightleftharpoons[\text{Str} \circ \text{Mod}]{\quad} \text{PThy}^{\text{op}}.$$

Proof. This follows from the analogous result on fibers over each $\mathcal{X}|\mathcal{A} \in \text{Prd}^R$ (Proposition 2.2.20 and Remark 2.2.23). $\square$

Remark 3.2.9. Let us explicate the behavior of the functor $\text{Str} : \text{Cat}_{//e_1^R} \rightarrow \text{PThy}^{\text{op}}$ defined above on maps. Suppose given a map $\psi : (\mathcal{X}|\mathcal{A}, \mathcal{Y}) \rightarrow (\mathcal{X}'|\mathcal{A}', \mathcal{Y}')$ in $\text{Cat}_{//e_1^R}$, consisting of a map $\rho : \mathcal{X}|\mathcal{A} \rightarrow \mathcal{X}'|\mathcal{A}'$ in Prd^R and a functor $\sigma : \mathcal{Y} \rightarrow \mathcal{Y}'$ making the square

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{\sigma} & \mathcal{Y}' \\ u \downarrow & & \downarrow u' \\ \mathcal{X} & \xrightarrow{\rho} & \mathcal{X}' \end{array}$$

commute, where u and u' are the structure maps. Let $\chi : \mathcal{A}^{\text{op}} \rightarrow \text{Str}_{\mathcal{A}}(\mathcal{Y})$ and $\chi' : (\mathcal{A}')^{\text{op}} \rightarrow \text{Str}_{\mathcal{A}'}(\mathcal{Y}')$ be the associated structural pretheories, which we recall are the factorizations of the functors

$$\begin{aligned} \mathcal{A}^{\text{op}} &\rightarrow \text{Fun}(\mathcal{Y}, \text{Spc}), & A &\mapsto \text{Map}_{\mathcal{X}}(A, u(-)) \\ (\mathcal{A}')^{\text{op}} &\rightarrow \text{Fun}(\mathcal{Y}', \text{Spc}), & A' &\mapsto \text{Map}_{\mathcal{X}'}(A', u'(-)) \end{aligned}$$

through their essential images. Letting $\lambda : \mathcal{X}' \rightarrow \mathcal{X}$ denote the left adjoint to ρ , we have natural equivalences

$$\text{Map}_{\mathcal{X}'}(A, u'\sigma(Y)) \simeq \text{Map}_{\mathcal{X}'}(A, \rho u(Y)) \simeq \text{Map}_{\mathcal{X}}(\lambda(A), u(Y))$$

for $A \in \mathcal{A}'$ and $Y \in \mathcal{Y}$. Hence, the restriction functor $\sigma^* : \text{Fun}(\mathcal{Y}', \text{Spc}) \rightarrow \text{Fun}(\mathcal{Y}, \text{Spc})$ induces a functor $\mu : \text{Str}_{\mathcal{A}'}(\mathcal{Y}') \rightarrow \text{Str}_{\mathcal{A}}(\mathcal{Y})$ that makes the following diagram commute:

$$\begin{array}{ccc} \mathcal{A}^{\text{op}} & \xleftarrow{\lambda} & (\mathcal{A}')^{\text{op}} \\ \chi \downarrow & & \downarrow \chi' \\ \text{Str}_{\mathcal{A}}(\mathcal{Y}) & \xleftarrow{\mu} & \text{Str}_{\mathcal{A}'}(\mathcal{Y}'). \end{array}$$

This defines a map $\phi : (\mathcal{X}|\mathcal{A}, \text{Str}_{\mathcal{A}}(\mathcal{Y})) \rightarrow (\mathcal{X}'|\mathcal{A}', \text{Str}_{\mathcal{A}'}(\mathcal{Y}'))$ in PThy (lying over the map λ in Prd^{L}), whose corresponding map in PThy^{op} (lying over the map ρ in Prd^{R}) is equivalent to the image of ψ under Str .

Let us note also that, if we subsequently apply the functor $\text{Mod} : \text{PThy}^{\text{op}} \rightarrow \text{Cat}_{\text{e}_1^{\text{R}}}$ to $\phi \simeq \text{Str}(\psi)$ and consider the unit transformation for the adjunction between Str and Mod , we obtain the following extension of our original commutative square of ∞ -categories:

$$\begin{array}{ccccc}
 \mathcal{Y} & \xrightarrow{\sigma} & \mathcal{Y}' & & \\
 \eta \searrow & & \eta' \searrow & & \\
 u \downarrow & \text{Mod}_{\mathcal{X}}(\text{Str}_{\mathcal{A}}(\mathcal{Y})) & \xrightarrow{\mu^*} & \text{Mod}_{\mathcal{X}'}(\text{Str}_{\mathcal{A}'}(\mathcal{Y}')) & \\
 v \nearrow & & u' \downarrow & & \\
 \mathcal{X} & \xrightarrow{\rho} & \mathcal{X}' & & \\
 & & v' \nearrow & &
 \end{array}$$

here v and v' denote the structure maps, and μ^* denotes restriction along μ (see Remark 3.2.4).

Let us take note of how the adjunction $\text{Str} : \text{Cat}_{\text{e}_1^{\text{R}}} \rightleftarrows \text{PThy}^{\text{op}} : \text{Mod}$ constructed above interacts with the cartesianness properties of the two ∞ -categories over Prd^{R} involved.

Remark 3.2.10. Recall that the projection functors $\text{PThy}^{\text{op}} \rightarrow \text{Prd}^{\text{R}}$ and $\text{Cat}_{\text{e}_1^{\text{R}}} \rightarrow \text{Prd}^{\text{R}}$ are bicartesian fibrations (Corollaries 3.1.9 and 3.1.13 and Remark 3.2.2). By [26, Proposition 7.3.2.6], the relative left adjoint functor $\text{Str} : \text{Cat}_{\text{e}_1^{\text{R}}} \rightarrow \text{PThy}^{\text{op}}$ preserves cocartesian maps and the relative right adjoint functor $\text{Mod} : \text{PThy}^{\text{op}} \rightarrow \text{Cat}_{\text{e}_1^{\text{R}}}$ preserves cartesian maps. In other words, for a presented adjunction $\lambda : \mathcal{X}'|\mathcal{A}' \rightleftarrows \mathcal{X}|\mathcal{A} : \rho$, we have commutative squares as follows:

$$\begin{array}{ccc}
 \text{Cat}/\mathcal{X} & \xrightarrow{\text{Str}_{\mathcal{A}}} & \text{PThy}_{\mathcal{A}}^{\text{op}} \\
 \rho_! \downarrow & & \downarrow \lambda^* = \rho_! \\
 \text{Cat}/\mathcal{X}' & \xrightarrow{\text{Str}_{\mathcal{A}'}} & \text{PThy}_{\mathcal{A}'}^{\text{op}}
 \end{array}
 \quad
 \begin{array}{ccc}
 \text{Cat}/\mathcal{X} & \xleftarrow{\text{Mod}_{\mathcal{X}}} & \text{PThy}_{\mathcal{A}}^{\text{op}} \\
 \rho^* \uparrow & & \uparrow \lambda_! \\
 \text{Cat}/\mathcal{X}' & \xleftarrow{\text{Mod}_{\mathcal{X}'}} & \text{PThy}_{\mathcal{A}'}^{\text{op}}
 \end{array}$$

(see Notations 3.1.10 and 3.1.14 for the right vertical arrows of these squares), the two squares related by passing to adjoints.

Remark 3.2.11. It follows from Proposition 2.2.20 that the right square in Remark 3.2.10, after restriction from pretheories to theories, factors into a commutative diagram as follows:

$$\begin{array}{ccccccc}
 \text{Cat}/\mathcal{X} & \xleftarrow{\quad} & \text{Cat}_{\mathcal{X}}^{\mathcal{A}\text{-th}} & \xleftarrow{\text{Mod}_{\mathcal{X}}} & \text{Thy}_{\mathcal{X}|\mathcal{A}}^{\text{op}} & \xleftarrow{\text{Str}_{\mathcal{A}} \text{Mod}_{\mathcal{X}}} & \text{Thy}_{\mathcal{A}}^{\text{op}} \\
 \rho^* \uparrow & & \rho^* \uparrow & & \uparrow \lambda_{!!} & & \uparrow \lambda_! \\
 \text{Cat}/\mathcal{X}' & \xleftarrow{\quad} & \text{Cat}_{\mathcal{X}'}^{\mathcal{A}'\text{-th}} & \xleftarrow{\text{Mod}_{\mathcal{X}'}} & \text{Thy}_{\mathcal{X}'|\mathcal{A}'}^{\text{op}} & \xleftarrow{\text{Str}_{\mathcal{A}'} \text{Mod}_{\mathcal{X}'}} & \text{Thy}_{\mathcal{A}'}^{\text{op}}
 \end{array}$$

(for the rightmost square here, see Remark 3.1.15). In other words, the restricted functor $\text{Mod} : \text{Thy}^{\text{op}} \rightarrow \text{Cat}_{\text{e}_1^{\text{R}}}$ also preserves cartesian maps over Prd^{R} .

We now describe a couple of special situations in which the compatibilities expressed in Remark 3.2.10 can be substantially refined.

Proposition 3.2.12. *Let $\lambda : \mathcal{X}'|\mathcal{A}' \rightleftarrows \mathcal{X}|\mathcal{A} : \rho$ be a presented adjunction. Assume that the functor $\rho : \mathcal{X} \rightarrow \mathcal{X}'$ is $\mathcal{A}'$ -theoretical and that the restricted functor $\lambda : \mathcal{A}' \rightarrow \mathcal{A}$ is essentially surjective. Then:*

1. $\lambda^{\text{op}} : (\mathcal{A}')^{\text{op}} \rightarrow \mathcal{A}^{\text{op}}$ is an $\mathcal{X}'|\mathcal{A}'$ -theory, and the functor $\lambda^* : \text{PThy}_{\mathcal{A}} \rightarrow \text{PThy}_{\mathcal{A}'}$ of Notation 3.1.10 induces equivalences

$$\text{PThy}_{\mathcal{A}} \simeq (\text{PThy}_{\mathcal{A}'})_{\mathcal{A}^{\text{op}}/}, \quad \text{Thy}_{\mathcal{A}} \simeq (\text{Thy}_{\mathcal{A}'})_{\mathcal{A}^{\text{op}}/}, \quad \text{Thy}_{\mathcal{X}|\mathcal{A}} \simeq (\text{Thy}_{\mathcal{X}'|\mathcal{A}'})_{\mathcal{A}^{\text{op}}/};$$

2. the left hand square in Remark 3.2.10 is horizontally right adjointable (equivalently the right hand square there is vertically left adjointable), giving a commutative square

$$\begin{array}{ccc} \text{Cat}_{/\mathcal{X}} & \xleftarrow{\text{Mod}_{\mathcal{X}}} & \text{PThy}_{\mathcal{A}}^{\text{op}} \\ \rho_! \downarrow & & \downarrow \lambda^* = \rho_! \\ \text{Cat}_{/\mathcal{X}'} & \xleftarrow{\text{Mod}_{\mathcal{X}'}} & \text{PThy}_{\mathcal{A}'}^{\text{op}} \end{array}$$

3. a functor $u : \mathcal{Y} \rightarrow \mathcal{X}$ is $\mathcal{A}$ -theorizable (resp. $\mathcal{A}$ -theoretical) if and only if the composition $\rho u : \mathcal{Y} \rightarrow \mathcal{X}'$ is $\mathcal{A}'$ -theorizable (resp. $\mathcal{A}'$ -theoretical), so that the functor $\rho_! : \text{Cat}_{/\mathcal{X}} \rightarrow \text{Cat}_{/\mathcal{X}'}$ induces equivalences

$$\text{Cat}_{/\mathcal{X}}^{\mathcal{A}\text{-thbl}} \simeq (\text{Cat}_{/\mathcal{X}'}^{\mathcal{A}'\text{-thbl}})_{/\mathcal{X}}, \quad \text{Cat}_{/\mathcal{X}}^{\mathcal{A}\text{-th}} \simeq (\text{Cat}_{/\mathcal{X}'}^{\mathcal{A}'\text{-th}})_{/\mathcal{X}};$$

4. the commutative square in 2 induces a commutative square of equivalences

$$\begin{array}{ccc} \text{Cat}_{/\mathcal{X}}^{\mathcal{A}\text{-th}} & \xleftarrow{\text{Mod}_{\mathcal{X}}} & \text{Thy}_{\mathcal{X}|\mathcal{A}}^{\text{op}} \\ \rho_! \downarrow & & \downarrow \lambda^* = \rho_! \\ (\text{Cat}_{/\mathcal{X}'}^{\mathcal{A}'\text{-th}})_{/\mathcal{X}} & \xleftarrow{\text{Mod}_{\mathcal{X}'}} & ((\text{Thy}_{\mathcal{X}'|\mathcal{A}'}^{\text{op}})_{\mathcal{A}^{\text{op}}})^{\text{op}}. \end{array}$$

Proof. Because $\lambda : \mathcal{A}' \rightarrow \mathcal{A}$ is essentially surjective, the image of an $\mathcal{A}$ -pretheory $\chi : \mathcal{A}^{\text{op}} \rightarrow \mathcal{T}$ under the restriction functor $\lambda^* : \text{PThy}_{\mathcal{A}} \rightarrow \text{PThy}_{\mathcal{A}'}$ is simply the composition $\chi \circ \lambda^{\text{op}} : (\mathcal{A}')^{\text{op}} \rightarrow \mathcal{T}$. In particular, $\lambda^{\text{op}} : (\mathcal{A}')^{\text{op}} \rightarrow \mathcal{A}$ is the restriction of $\text{id}_{\mathcal{A}^{\text{op}}} : \mathcal{A}^{\text{op}} \rightarrow \mathcal{A}^{\text{op}}$. Since restriction carries $\mathcal{X}|\mathcal{A}$ -theories to $\mathcal{X}'|\mathcal{A}'$ -theories (Proposition 3.1.8.2) and $\text{id}_{\mathcal{A}^{\text{op}}}$ is a $\mathcal{X}|\mathcal{A}$ -theory, this proves that $\lambda^{\text{op}} : (\mathcal{A}')^{\text{op}} \rightarrow \mathcal{A}^{\text{op}}$ is an $\mathcal{X}'|\mathcal{A}'$ -theory.

Precomposition with λ^{op} determines an equivalence $\text{Cat}_{\mathcal{A}^{\text{op}}/\mathcal{X}} \simeq (\text{Cat}_{(\mathcal{A}')^{\text{op}}/\mathcal{X}})_{\mathcal{A}^{\text{op}}/\mathcal{X}}$, so to finish proving 1, it remains only to show that a functor $\chi : \mathcal{A}^{\text{op}} \rightarrow \mathcal{T}$ is an $\mathcal{A}$ -theory (resp. $\mathcal{X}|\mathcal{A}$ -theory) if $\chi \circ \lambda^{\text{op}} : (\mathcal{A}')^{\text{op}} \rightarrow \mathcal{T}$ is an $\mathcal{A}'$ -theory (resp. $\mathcal{X}'|\mathcal{A}'$ -theory). The local smallness condition on $\mathcal{T}$ doesn't change and χ must be essentially surjective if $\chi \circ \lambda^{\text{op}}$ is. Finally, χ is an $\mathcal{X}|\mathcal{A}$ -object if $\chi \lambda^{\text{op}}$ is an $\mathcal{X}'|\mathcal{A}'$ -object, because the square

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{j_{\mathcal{X}|\mathcal{A}}} & \mathcal{P}(\mathcal{A}) \\ \rho \downarrow & & \downarrow \lambda^* \\ \mathcal{X}' & \xrightarrow{j_{\mathcal{X}'|\mathcal{A}'}} & \mathcal{P}(\mathcal{A}') \end{array}$$

is cartesian: this is the hypothesis that ρ is $\mathcal{A}'$ -theoretical.

The cartesianness of the above square (together with unwinding definitions) also gives 2. From 2 we immediately get 4 (the lower horizontal arrow in the square there is an equivalence by 1 and the vertical arrows are equivalences by Remark 2.2.23, so the upper horizontal arrow is also an equivalence). Finally, the $\mathcal{A}$ -theorizable case of 3 follow from 1 and the commutativity of the left hand square in Remark 3.2.10, while in the $\mathcal{A}$ -theoretical case they follow from 4. $\square$

Remark 3.2.13. In the situation of Proposition 3.2.12, we may combine the left hand square of Remark 3.2.10 and the square of Proposition 3.2.12.4 to get the following commutative square, which identifies $\mathcal{A}$ -theorization over $\mathcal{X}$ with $\mathcal{A}'$ -theorization over $\mathcal{X}'$:

$$\begin{array}{ccc} \text{Cat}_{/\mathcal{X}}^{\mathcal{A}\text{-thbl}} & \xrightarrow{(-)^{\mathcal{A}\text{-th}/\mathcal{X}}} & \text{Cat}_{/\mathcal{X}}^{\mathcal{A}\text{-th}} \\ \rho_! \downarrow & & \downarrow \rho_! \\ (\text{Cat}_{/\mathcal{X}'}^{\mathcal{A}'\text{-thbl}})_{/\mathcal{X}} & \xrightarrow{(-)^{\mathcal{A}'\text{-th}/\mathcal{X}'}} & (\text{Cat}_{/\mathcal{X}'}^{\mathcal{A}'\text{-th}})_{/\mathcal{X}}. \end{array}$$

Proposition 3.2.14. Let $\mathcal{X}|\mathcal{A}$ be a presented ∞ -category, and let $\mathcal{A}' \subseteq \mathcal{A}$ be a full subcategory such that every object of $\mathcal{A}$ is a retract of an object of $\mathcal{A}'$, so that $\mathcal{A}'$ is also dense in $\mathcal{X}$ (Remark 2.1.2) and hence the identity functor of $\mathcal{X}$ defines a presented adjunction $\lambda = \text{id}_{\mathcal{X}} : \mathcal{X}|\mathcal{A}' \rightleftarrows \mathcal{X}|\mathcal{A} : \text{id}_{\mathcal{X}} = \rho$. Then:

1. the functor $\text{id}_\chi^* : \text{Thy}_{\mathcal{X}|\mathcal{A}} \rightarrow \text{Thy}_{\mathcal{X}|\mathcal{A}'}$ of Notation 3.1.10 is an equivalence;
2. the left hand square in Remark 3.2.10 is horizontally right adjointable (equivalently the right hand square there is vertically left adjointable), giving a commutative square

$$\begin{array}{ccc} \text{Cat}/\chi & \xleftarrow{\text{Mod}_\chi} & \text{PThy}_{\mathcal{A}}^{\text{op}} \\ \text{id} \downarrow & & \downarrow \text{id}_\chi^* = (\text{id}_\chi)_! \\ \text{Cat}/\chi & \xleftarrow{\text{Mod}_\chi} & \text{PThy}_{\mathcal{A}'}^{\text{op}}; \end{array}$$

3. a functor $u : \mathcal{Y} \rightarrow \mathcal{X}$ is $\mathcal{A}$ -theorizable (resp. $\mathcal{A}$ -theoretical) if and only if it is $\mathcal{A}'$ -theorizable (resp. $\mathcal{A}'$ -theoretical), and we have a commutative square

$$\begin{array}{ccc} \text{Cat}_{/\chi}^{\mathcal{A}\text{-thbl}} & \xrightarrow{(-)^{\mathcal{A}\text{-th}/\chi}} & \text{Cat}_{/\chi}^{\mathcal{A}\text{-th}} \\ \text{id} \downarrow & & \downarrow \text{id} \\ \text{Cat}_{/\chi}^{\mathcal{A}'\text{-thbl}} & \xrightarrow{(-)^{\mathcal{A}'\text{-th}/\chi}} & \text{Cat}_{/\chi}^{\mathcal{A}'\text{-th}}, \end{array}$$

identifying $\mathcal{A}$ -theorization with $\mathcal{A}'$ -theorization.

Proof. We begin with 2. Let $\chi : \mathcal{A}^{\text{op}} \rightarrow \mathcal{T}$ be an $\mathcal{A}$ -pretheory and let $\chi' : (\mathcal{A}')^{\text{op}} \rightarrow \mathcal{T}'$ be its restriction. The claim is that the functor $\text{Mod}_\chi(\mathcal{T}) \rightarrow \text{Mod}_\chi(\mathcal{T}')$ given by restriction along the inclusion $\mathcal{T}' \hookrightarrow \mathcal{T}$ is an equivalence. This follows from the fact that the restriction functors $\text{Fun}(\mathcal{A}, \text{Spc}) \rightarrow \text{Fun}(\mathcal{A}', \text{Spc})$ and $\text{Fun}(\mathcal{T}, \text{Spc}) \rightarrow \text{Fun}(\mathcal{T}', \text{Spc})$ are equivalences, since every object in $\mathcal{A}$ is a retract of an object of $\mathcal{A}'$, which implies that also every object in $\mathcal{T}$ is a retract of an object of $\mathcal{T}'$.

We now prove 1. From 2 and the fact that $\text{Mod}_\chi : \text{Thy}_{\mathcal{X}|\mathcal{A}} \rightarrow \text{Cat}/\chi$ and $\text{Mod}_\chi : \text{Thy}_{\mathcal{X}|\mathcal{A}'} \rightarrow \text{Cat}/\chi$ are fully faithful, we deduce that $\text{id}_\chi^* : \text{Thy}_{\mathcal{X}|\mathcal{A}} \rightarrow \text{Thy}_{\mathcal{X}|\mathcal{A}'}$ is fully faithful. Similarly, the commutative diagram in Remark 3.2.11 implies that the left adjoint $(\text{id}_\chi)_! : \text{Thy}_{\mathcal{X}|\mathcal{A}'} \rightarrow \text{Thy}_{\mathcal{X}|\mathcal{A}}$ is fully faithful. Hence the two adjoints must be equivalences.

Finally, we address 3. The statement for theoreticity is immediate from 1 and 2. For theorizability, it suffices, by virtue of the left hand square in Remark 3.2.10, to show that an $\mathcal{A}$ -pretheory $\chi : \mathcal{A}^{\text{op}} \rightarrow \mathcal{T}$ is an $\mathcal{A}$ -theory (i.e. $\mathcal{T}$ is locally small) if and only if its restriction to an $\mathcal{A}'$ -pretheory $\chi' : (\mathcal{A}')^{\text{op}} \rightarrow \mathcal{T}'$ is an $\mathcal{A}'$ -theory (i.e. $\mathcal{T}'$ is locally small). This follows from the fact that, for $T_0, T_1 \in \mathcal{T}$, the essential smallness of $\text{Map}_{\mathcal{T}}(T_0, T_1)$ is stable under taking retracts of both T_0 and T_1 . Lastly, we get the claimed commutative square by combining the left hand square in Remark 3.2.10 and the square of 2. $\square$

We end this section by applying some of our observations to tie up a minor point from Remark 2.2.25.

Proposition 3.2.15. *Let $\mathcal{X}|\mathcal{A}$ be a presented ∞ -category and let $\mathcal{Y} \in \text{Cat}/\chi$. Then $\mathcal{Y}$ admits an $\mathcal{A}$ -theorization (Definition 2.2.24) if and only if $\mathcal{Y}$ is $\mathcal{A}$ -theorizable (Definition 2.2.15).*

Proof. The “if” direction was observed already in Remark 2.2.25; let us prove the converse. Assume that $\mathcal{Y}$ admits an $\mathcal{A}$ -theorization. This is equivalent to $\text{Mod}_\chi : \text{Thy}_{\mathcal{X}|\mathcal{A}}^{\text{op}} \rightarrow \text{Cat}/\chi$ admitting a left adjoint $\mathcal{T}_?$ at $\mathcal{Y}$. By [26, Proof of Proposition 7.3.2.6], then $(\mathcal{X}|\mathcal{A}, \mathcal{T}_?)$ is a left adjoint of the functor $\text{Mod} : \text{Thy}^{\text{op}} \rightarrow \text{Cat}_{/\text{eR}}$ at $(\mathcal{X}|\mathcal{A}, \mathcal{Y})$, because this functor preserves cartesian maps over Prd^{R} (Remark 3.2.11). On the other hand, we have that $(\mathcal{X}|\mathcal{A}, \text{Str}_{\mathcal{A}}(\mathcal{Y}))$ is a left adjoint of $\text{Mod} : \text{PThy}^{\text{op}} \rightarrow \text{Cat}_{/\text{eR}}$ at $(\mathcal{X}|\mathcal{A}, \mathcal{Y})$ (Proposition 3.2.5). It follows that we have a canonical map of $\mathcal{A}$ -pretheories $\phi : \mathcal{T}_? \rightarrow \text{Str}_{\mathcal{A}}(\mathcal{Y})$, inducing an equivalence

$$\text{Map}_{\text{Thy}}((\mathcal{X}'|\mathcal{A}', \mathcal{T}'), (\mathcal{X}|\mathcal{A}, \mathcal{T}_?)) \rightarrow \text{Map}_{\text{PThy}}((\mathcal{X}'|\mathcal{A}', \mathcal{T}'), (\mathcal{X}|\mathcal{A}, \text{Str}_{\mathcal{A}}(\mathcal{Y})))$$

for any $(\mathcal{X}'|\mathcal{A}', \mathcal{T}') \in \text{Thy}$. It's enough to prove that ϕ is an equivalence.

We may let $\mathcal{A}'$ be the discrete category $\{0, 1\}$, let $\mathcal{X}' := \mathcal{P}(\mathcal{A}')$, let $\mathcal{T}' := \Delta^1$, and let $\chi' : (\mathcal{A}')^{\text{op}} = \mathcal{A}' \rightarrow \mathcal{T}'$ be the inclusion: allowing ourselves to identify $\mathcal{A}'$ with its image in $\mathcal{X}'$ under the Yoneda embedding, we have that χ' is an $\mathcal{X}'|\mathcal{A}'$ -theory. In this case, the above map being an equivalence means that, for any

$A_0, A_1 \in \mathcal{A}$, and with $\chi : \mathcal{A}^{\text{op}} \rightarrow \text{Str}_{\mathcal{A}}(\mathcal{Y})$ and $\chi_{\mathcal{T}} : \mathcal{A}^{\text{op}} \rightarrow \mathcal{T}_{\mathcal{T}}$ denoting the structure maps, the map

$$\text{Map}_{\mathcal{T}}(\chi_{\mathcal{T}}(A_0), \chi_{\mathcal{T}}(A_1)) \rightarrow \text{Map}_{\text{Str}_{\mathcal{A}}(\mathcal{Y})}(\chi(A_0), \chi(A_1))$$

induced by ϕ is an equivalence. That is, ϕ is full faithful, which implies that it's an equivalence (any map of $\mathcal{A}$ -pretheories being essentially surjective). $\square$

§3.3. Covariance. The previous section concerned the completely general contravariant functoriality of the constructions $\mathcal{T} \mapsto \text{Mod}_{\mathcal{X}}(\mathcal{T})$ and $\mathcal{Y} \mapsto \text{Str}_{\mathcal{A}}(\mathcal{Y})$. In some special circumstances, these constructions additionally exhibit a covariant functoriality. In this section, we'll formulate the basic elements of this covariance, and then describe some cases where it occurs.

We first formulate a type of covariant functoriality that may occur for ∞ -categories of models.

Construction 3.3.1. Let $\phi : (\mathcal{X}'|\mathcal{A}', \mathcal{T}') \rightarrow (\mathcal{X}|\mathcal{A}, \mathcal{T})$ be a map in Thy , consisting of a map $\lambda : \mathcal{X}'|\mathcal{A}' \rightarrow \mathcal{X}|\mathcal{A}$ in Prd^{L} , an $\mathcal{X}|\mathcal{A}$ -theory $\chi : \mathcal{A}^{\text{op}} \rightarrow \mathcal{T}$, an $\mathcal{X}'|\mathcal{A}'$ -theory $\chi' : (\mathcal{A}')^{\text{op}} \rightarrow \mathcal{T}'$, and functor $\mu : \mathcal{T}' \rightarrow \mathcal{T}$ making the following square commute:

$$\begin{array}{ccc} (\mathcal{A}')^{\text{op}} & \xrightarrow{\lambda} & \mathcal{A}^{\text{op}} \\ \chi' \downarrow & & \downarrow \chi \\ \mathcal{T}' & \xrightarrow{\mu} & \mathcal{T}. \end{array}$$

Recall from Remark 3.2.4 that this induces a commutative square

$$(3.3.2) \quad \begin{array}{ccc} \text{Mod}_{\mathcal{X}'}(\mathcal{T}') & \xleftarrow{\mu^*} & \text{Mod}_{\mathcal{X}}(\mathcal{T}) \\ u' \downarrow & & \downarrow u \\ \mathcal{X}' & \xleftarrow{\rho} & \mathcal{X} \end{array}$$

where ρ is right adjoint to λ , the functor μ^* is given by restriction along μ , and u and u' are the structure maps. Note that this square lies in Pr^{R} . Letting $\mu_{\dagger} : \text{Mod}_{\mathcal{X}'}(\mathcal{T}') \rightarrow \text{Mod}_{\mathcal{X}}(\mathcal{T})$ denote the left adjoint to μ^* , we get an induced natural transformation $\beta : \lambda \circ u' \rightarrow u \circ \mu_{\dagger}$.

Remark 3.3.3. The functor $\mu_{\dagger}$ in Construction 3.3.1 can be characterized as follows: it is left Kan extended along the Yoneda embedding $j_{\mathcal{T}'}^{\vee} : \mathcal{T}' \rightarrow \text{Mod}_{\mathcal{X}'}(\mathcal{T}')$ and makes the following square commute:

$$\begin{array}{ccc} \mathcal{T}' & \xrightarrow{\mu} & \mathcal{T} \\ j_{\mathcal{T}'}^{\vee} \downarrow & & \downarrow j_{\mathcal{T}}^{\vee} \\ \text{Mod}_{\mathcal{X}'}(\mathcal{T}') & \xrightarrow{\mu_{\dagger}} & \text{Mod}_{\mathcal{X}}(\mathcal{T}). \end{array}$$

Definition 3.3.4. In the situation of Construction 3.3.1, we say that the functor μ is *model adjointable over λ* if the natural transformation β is an equivalence, i.e. the square (3.3.2) is horizontally left adjointable, giving a commutative square

$$\begin{array}{ccc} \text{Mod}_{\mathcal{X}'}(\mathcal{T}') & \xrightarrow{\mu_{\dagger}} & \text{Mod}_{\mathcal{X}}(\mathcal{T}) \\ u' \downarrow & & \downarrow u \\ \mathcal{X}' & \xrightarrow{\lambda} & \mathcal{X}. \end{array}$$

Conversely, we may consider the following form of covariant functoriality for structural pretheories.

Construction 3.3.5. Suppose given a map $\lambda : \mathcal{X}'|\mathcal{A}' \rightarrow \mathcal{X}|\mathcal{A}$ in Prd^{L} and functors $u : \mathcal{Y} \rightarrow \mathcal{X}$ and $u' : \mathcal{Y}' \rightarrow \mathcal{X}'$. Let $\nu : \mathcal{Y}' \rightarrow \mathcal{Y}$ be a functor equipped with a natural transformation $\beta : \lambda \circ u' \rightarrow u \circ \nu$, which we may depict

diagrammatically as follows:

$$\begin{array}{ccc} \mathcal{Y}' & \xrightarrow{\nu} & \mathcal{Y} \\ u' \downarrow & \nearrow \beta & \downarrow u \\ \mathcal{X}' & \xrightarrow{\lambda} & \mathcal{X}. \end{array}$$

Let $\chi : \mathcal{A}^{\text{op}} \rightarrow \text{Str}_{\mathcal{A}}(\mathcal{Y}) \subseteq \text{Fun}(\mathcal{Y}, \text{Spc})$ and $\chi' : (\mathcal{A}')^{\text{op}} \rightarrow \text{Str}_{\mathcal{A}'}(\mathcal{Y}') \subseteq \text{Fun}(\mathcal{Y}', \text{Spc})$ be the structural pretheories, and note that we have a restriction functor $\nu^* : \text{Fun}(\mathcal{Y}, \text{Spc}) \rightarrow \text{Fun}(\mathcal{Y}', \text{Spc})$. We define the natural transformation

$$\gamma : \chi' \rightarrow \nu^* \circ \chi \circ \lambda$$

to be that given by the composition

$$\chi'(A)(Y) \simeq \text{Map}_{\mathcal{X}'}(A, u'(Y)) \xrightarrow{\lambda} \text{Map}_{\mathcal{X}}(\lambda(A), \lambda u'(Y)) \xrightarrow{\beta} \text{Map}_{\mathcal{X}}(\lambda(A'), u\nu(Y)) \simeq \chi(\lambda(A))(\nu(Y))$$

for $A \in (\mathcal{A}')^{\text{op}}$ and $Y \in \mathcal{Y}'$.

Definition 3.3.6. In the situation of Construction 3.3.5, we say that the pair (ν, β) (or simply the functor ν , when β may be understood from context) *respects structure over λ* if, for each $A \in \mathcal{A}'$, the map $\gamma : \chi'(A) \rightarrow \nu^*(\chi(\lambda(A)))$ exhibits $\chi(\lambda(A))$ as the left Kan extension of $\chi'(A)$ along ν . If this holds, then, by the functoriality of left Kan extension, it gives us a commutative diagram

$$\begin{array}{ccc} (\mathcal{A}')^{\text{op}} & \xrightarrow{\lambda} & \mathcal{A}^{\text{op}} \\ \chi' \downarrow & & \downarrow \chi \\ \text{Str}_{\mathcal{A}'}(\mathcal{Y}') & \xrightarrow{\nu_!} & \text{Str}_{\mathcal{A}}(\mathcal{Y}), \end{array}$$

defining a map $(\mathcal{X}'|\mathcal{A}', \text{Str}_{\mathcal{A}'}(\mathcal{Y}')) \rightarrow (\mathcal{X}|\mathcal{A}, \text{Str}_{\mathcal{A}}(\mathcal{Y}))$ in PThy over the map λ in Prd^{L} .

Let us now examine one general situation (a concrete case of which will arise in the subsequent sections of the paper) in which we may observe the phenomena of Definitions 3.3.4 and 3.3.6.

Definition 3.3.7. Let $\lambda : \mathcal{X}'|\mathcal{A}' \rightleftarrows \mathcal{X}|\mathcal{A} : \rho$ be a map in Prd^{L} , and let s denote the associated commutative square

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{j_{\mathcal{X}|\mathcal{A}}} & \mathcal{P}(\mathcal{A}) \\ \rho \downarrow & & \downarrow \lambda^* \\ \mathcal{X}' & \xrightarrow{j_{\mathcal{X}'|\mathcal{A}'}} & \mathcal{P}(\mathcal{A}'), \end{array}$$

where ρ is the right adjoint to λ . We call λ :

1. an *embedding* if the underlying functor $\lambda : \mathcal{X}' \rightarrow \mathcal{X}$ is fully faithful;
2. a *strict embedding* if it is an embedding in Prd^{L} and s is vertically left adjointable.

Remark 3.3.8. Let $\lambda : \mathcal{X}'|\mathcal{A}' \rightarrow \mathcal{X}|\mathcal{A}$ be a strict embedding in Prd^{L} , so that we have a commutative square

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{j_{\mathcal{X}|\mathcal{A}}} & \mathcal{P}(\mathcal{A}) \\ \lambda \uparrow & & \uparrow \lambda_! \\ \mathcal{X}' & \xrightarrow{j_{\mathcal{X}'|\mathcal{A}'}} & \mathcal{P}(\mathcal{A}'), \end{array}$$

where $\lambda_!$ denotes left Kan extension along λ . This square is in fact cartesian; in other words, an object $X \in \mathcal{X}$ lies in the essential image of the embedding $\lambda : \mathcal{X}' \rightarrow \mathcal{X}$ if and only if $j_{\mathcal{X}|\mathcal{A}}$ is left Kan extended along $\lambda : \mathcal{A}' \rightarrow \mathcal{A}$. The “only if” direction is the commutativity of the above square, i.e. the hypothesized adjointability of the square s in Definition 3.3.7; the “if” direction follows from considering the square obtained from s by passing to left adjoints. In this sense, λ is determined by $\mathcal{X}|\mathcal{A}$ and the restricted embedding $\lambda : \mathcal{A}' \rightarrow \mathcal{A}$.

Proposition 3.3.9. *Let $\lambda : \mathcal{X}'|\mathcal{A}' \rightarrow \mathcal{X}|\mathcal{A}$ be an embedding in Prd^{L} ; and let $\mathcal{Y}' \in \text{Cat}_{/\mathcal{X}'}$, which we also regard as an ∞ -category over $\mathcal{X}$ by composition with λ . Then:*

1. *the identity functor $\text{id} : \mathcal{Y}' \rightarrow \mathcal{Y}'$ respects structure over λ ;*
2. *the induced map $(\mathcal{X}'|\mathcal{A}', \text{Str}_{\mathcal{A}'}(\mathcal{Y}')) \rightarrow (\mathcal{X}|\mathcal{A}, \text{Str}_{\mathcal{A}}(\mathcal{Y}'))$ in PThy is a cartesian lift of λ , i.e. identifies $\text{Str}_{\mathcal{A}'}(\mathcal{Y}')$ with the restriction of $\text{Str}_{\mathcal{A}}(\mathcal{Y})$ along λ .*

Proof. The two assertions amount to the claim that the map γ of Construction 3.3.5 is an equivalence in this situation. This follows from the full faithfulness of λ (and the natural transformation β , kept implicit in the statement, being an identity map). $\square$

Definition 3.3.10. Let $\lambda : \mathcal{X}'|\mathcal{A}' \rightarrow \mathcal{X}|\mathcal{A}$ be an embedding in Prd^{L} , let $\chi : \mathcal{A}^{\text{op}} \rightarrow \mathcal{T}$ be an $\mathcal{X}|\mathcal{A}$ -theory, and let $\chi' : (\mathcal{A}')^{\text{op}} \rightarrow \mathcal{T}'$ be the restriction of $\mathcal{T}$ along λ (Construction 3.1.7). We say that $\mathcal{T}$ is λ -consistent if, for each $T \in \mathcal{T}'$, the object $\chi_T \in \mathcal{X}$ lies in the essential image of λ . We write $\text{Thy}_{\mathcal{X}|\mathcal{A}}^{\lambda\text{-cons}}$ for the full subcategory of $\text{Thy}_{\mathcal{X}|\mathcal{A}}$ spanned by the λ -consistent $\mathcal{X}|\mathcal{A}$ -theories.

Proposition 3.3.11. *Suppose that we are in the situation of Definition 3.3.10, and let $\mu : \mathcal{T}' \hookrightarrow \mathcal{T}$ denote the inclusion functor. Then the following statements hold:*

1. *Assume that μ is model adjointable over λ . Then the resulting commutative square*

$$\begin{array}{ccc} \text{Mod}_{\mathcal{X}'}(\mathcal{T}') & \xrightarrow{\mu_{\dagger}} & \text{Mod}_{\mathcal{X}}(\mathcal{T}) \\ u' \downarrow & & \downarrow u \\ \mathcal{X}' & \xrightarrow{\lambda} & \mathcal{X} \end{array}$$

(as in Definition 3.3.4) is cartesian, and $\mathcal{T}$ is λ -consistent.

2. *Assume that λ is a strict embedding in Prd^{L} and that $\mathcal{T}$ is λ -consistent. Then μ is model adjointable over λ , and the commutative diagram*

$$\begin{array}{ccc} \mathcal{P}^{\vee}(\mathcal{T}') & \xleftarrow{\mu^*} & \mathcal{P}^{\vee}(\mathcal{T}) \\ \uparrow & & \uparrow \\ \text{Mod}_{\mathcal{X}'}(\mathcal{T}') & \xleftarrow{\mu^*} & \text{Mod}_{\mathcal{X}}(\mathcal{T}), \end{array}$$

is horizontally left adjointable, so that $\mu_{\dagger}$ identifies with a restriction of the left Kan extension functor $\mu_{!} : \mathcal{P}^{\vee}(\mathcal{T}') \rightarrow \mathcal{P}^{\vee}(\mathcal{T})$.

Lemma 3.3.12. *Suppose given a commutative square s :*

$$\begin{array}{ccc} \mathcal{Y}' & \xleftarrow{\sigma} & \mathcal{Y} \\ u' \downarrow & & \downarrow u \\ \mathcal{X}' & \xleftarrow{\rho} & \mathcal{X} \end{array}$$

in Pr^{R} , and suppose that s is horizontally left adjointable, so that we have a commutative square s' :

$$\begin{array}{ccc} \mathcal{Y}' & \xrightarrow{\nu} & \mathcal{Y} \\ u' \downarrow & & \downarrow u \\ \mathcal{X}' & \xrightarrow{\lambda} & \mathcal{X}, \end{array}$$

λ being the left adjoint to ρ and ν being the left adjoint to σ . Then:

1. *if u' is conservative and λ is fully faithful, ν is fully faithful;*
2. *if the conditions of 1 hold and moreover u is conservative, s' is cartesian.*

Proof. Assume that u' is conservative and that λ is fully faithful. Let $Y' \in \mathcal{Y}'$. To prove 1, it suffices to show that the unit map $\eta : Y' \rightarrow \sigma\nu(Y')$ is an equivalence. Since u' is conservative, it further suffices to

show that $u'(\eta) : u'(Y') \rightarrow u'\sigma\nu(Y')$ is an equivalence. We have equivalences

$$u'\sigma\nu(Y') \simeq \rho u\nu(Y') \simeq \rho \lambda u'(Y'),$$

under which $u'(\eta)$ identifies with the unit map $\bar{\eta} : u'(Y') \rightarrow \rho \lambda u'(Y')$, which is an equivalence because λ is fully faithful.

Now assume that u is also conservative. Suppose given $Y \in \mathcal{Y}$ such that $u(Y)$ lies in the essential image of λ . To prove 2, it suffices (having proved 1) to show that Y lies in the essential image of ν . This is equivalent to the counit map $\epsilon : \nu\sigma(Y) \rightarrow Y$ being an equivalence, which by the conservativity of u is equivalent to $u(\epsilon) : u\nu\sigma(Y) \rightarrow u(Y)$ being an equivalence. We have equivalences

$$u\nu\sigma(Y) \simeq \lambda u'\sigma(Y) \simeq \lambda \rho u(Y),$$

under which $u(\epsilon)$ identifies with the counit map $\bar{\epsilon} : \lambda \rho u(Y) \rightarrow u(Y)$, which is an equivalence because $u(Y)$ lies in the essential image of λ . $\square$

Proof of Proposition 3.3.11. The cartesianness claim in 1 follows from Lemma 3.3.12. For the other claims, we consider the commutative diagram

$$\begin{array}{ccccc} & & \mathcal{P}^\vee(\mathcal{T}') & \xleftarrow{\mu^*} & \mathcal{P}^\vee(\mathcal{T}) \\ & \nearrow i' & \downarrow & & \nearrow i \\ \text{Mod}_{\mathcal{X}'}(\mathcal{T}') & \xleftarrow{(\chi')^*} & \text{Mod}_{\mathcal{X}}(\mathcal{T}) & & \downarrow \chi^* \\ & \downarrow (x')^* & \downarrow \lambda^* & & \downarrow \chi^* \\ & \mathcal{P}(\mathcal{A}') & \xleftarrow{\lambda^*} & \mathcal{P}(\mathcal{A}) & \\ & \downarrow u' & \downarrow u & & \downarrow \chi^* \\ \mathcal{X}' & \xleftarrow{\rho} & \mathcal{X} & & \end{array}$$

in which: ρ is the right adjoint to λ ; u and u' are the structure maps; and i and i' are the inclusion functors. As in Construction 3.3.1 and Definition 3.3.4, the front square induces a natural transformation $\beta : \lambda \circ u' \rightarrow u \circ \mu_!$, and the condition that μ be model adjointable over λ means that β is an equivalence.

For $T \in \mathcal{T}$, we have a natural equivalence $\chi_T \simeq u(j_{\mathcal{T}}^\vee(T))$; and for $T \in \mathcal{T}'$, we have natural equivalences

$$\chi'_T \simeq u'(j_{\mathcal{T}'}^\vee(T)) \simeq u'\mu^*(j_{\mathcal{T}}^\vee(T)) \simeq \rho u(j_{\mathcal{T}}^\vee(T)) \simeq \rho(\chi_T).$$

It follows that the condition that $\mathcal{T}$ be λ -consistent is equivalent to the restriction of β along the Yoneda embedding $j_{\mathcal{T}'}^\vee : \mathcal{T}' \rightarrow \text{Mod}_{\mathcal{X}'}(\mathcal{T}')$ being an equivalence. This proves the implication in 1.

Now let us prove 2. Let $\bar{\beta} : \lambda_! \circ (\chi')^* \rightarrow \chi^* \circ \mu_!$ be the natural transformation induced by the back square of the diagram above, where $\lambda_! : \mathcal{P}(\mathcal{A}') \rightarrow \mathcal{P}(\mathcal{A})$ and $\mu_! : \mathcal{P}^\vee(\mathcal{T}') \rightarrow \mathcal{P}^\vee(\mathcal{T})$ are the left Kan extension functors. If λ is a strict embedding in Prd^L , i.e. the lower square in the diagram above is horizontally left adjointable, then $\mathcal{T}$ being λ -consistent is furthermore equivalent to the restriction of $\bar{\beta}$ to $\mathcal{T}'$ being an equivalence. This condition in fact implies that $\bar{\beta}$ is an equivalence, since its source and target preserve colimits and hence are left Kan extended from $\mathcal{T}'$. If $\bar{\beta}$ is an equivalence, then $\mu_!$ carries models of $\mathcal{T}'$ in $\mathcal{X}'$ to models of $\mathcal{T}$ in $\mathcal{X}$, giving the horizontal left adjointability of the claim (i.e. of the upper square in the diagram above); from this we then deduce that β is also an equivalence, completing the proof. $\square$

For the final part of this section, we turn to the following construction, in particular exhibiting its connection to Definition 3.3.10 and Proposition 3.3.11.

Construction 3.3.13. Let $\lambda : \mathcal{X}'|\mathcal{A}' \rightarrow \mathcal{X}|\mathcal{A}$ be a map in Prd^L . Consider the composition

$$\text{PThy}_{\mathcal{A}'} \xrightarrow{\text{Mod}_{\mathcal{X}'}} \text{Cat}_{/\mathcal{X}'} \xrightarrow{\lambda_!} \text{Cat}_{/\mathcal{X}} \xrightarrow{\text{Str}_{\mathcal{A}}} \text{PThy}_{\mathcal{A}}.$$

This restricts to a functor from $\text{Thy}_{\mathcal{X}'|\mathcal{A}'}$ to $\text{Thy}_{\mathcal{X}|\mathcal{A}}$: for T' an $\mathcal{X}'|\mathcal{A}'$ -theory, the structure map $\text{Mod}_{\mathcal{X}'}(\mathcal{T}') \rightarrow \mathcal{X}'$ is accessible, so its composition with λ is too, and hence the latter is $\mathcal{A}$ -theorizable by Example 2.2.17. We denote the restricted functor by

$$\lambda_{**} : \text{Thy}_{\mathcal{X}'|\mathcal{A}'} \rightarrow \text{Thy}_{\mathcal{X}|\mathcal{A}},$$

and we refer to it as *right pushforward along λ* .

Remark 3.3.14. The right pushforward functor defined above may be compared to the left pushforward functor $\lambda_{!!} : \text{Thy}_{\mathcal{X}'|\mathcal{A}'} \rightarrow \text{Thy}_{\mathcal{X}|\mathcal{A}}$ of Remark 3.1.15, which by Remark 3.2.11 identifies with the composition

$$\text{Thy}_{\mathcal{X}'|\mathcal{A}'} \xrightarrow{\text{Mod}_{\mathcal{X}'}} \text{Cat}_{/\mathcal{X}'} \xrightarrow{\rho^*} \text{Cat}_{/\mathcal{X}} \xrightarrow{\text{Str}_{\mathcal{A}}} \text{Thy}_{\mathcal{X}|\mathcal{A}},$$

where $\rho : \mathcal{X} \rightarrow \mathcal{X}'$ is the right adjoint to λ .

Definition 3.3.15. Let $\lambda : \mathcal{X}'|\mathcal{A}' \rightarrow \mathcal{X}|\mathcal{A}$ be a strict embedding in Prd^{L} . We say that an $\mathcal{X}|\mathcal{A}$ -theory $\chi : \mathcal{A}^{\text{op}} \rightarrow \mathcal{T}$ is λ -*extended* if it is left Kan extended along the embedding $\lambda : (\mathcal{A}')^{\text{op}} \rightarrow \mathcal{A}^{\text{op}}$.

Proposition 3.3.16. Let $\lambda : \mathcal{X}'|\mathcal{A}' \rightarrow \mathcal{X}|\mathcal{A}$ be a strict embedding in Prd^{L} . Then:

1. the functor $\lambda_{**} : \text{Thy}_{\mathcal{X}'|\mathcal{A}'} \rightarrow \text{Thy}_{\mathcal{X}|\mathcal{A}}$ is fully faithful, and its essential image consists of those $\mathcal{X}|\mathcal{A}$ -theories that are both λ -consistent and λ -extended;
2. we have an adjunction $\lambda^* : \text{Thy}_{\mathcal{X}|\mathcal{A}}^{\lambda\text{-cons}} \rightleftarrows \text{Thy}_{\mathcal{X}'|\mathcal{A}'} : \lambda_{**}$.

Lemma 3.3.17. Let $\lambda : \mathcal{X}'|\mathcal{A}' \rightarrow \mathcal{X}|\mathcal{A}$ be a strict embedding in Prd^{L} , let $u' : \mathcal{Y}' \rightarrow \mathcal{X}'$ be a functor such that the composition $\lambda u' : \mathcal{Y}' \rightarrow \mathcal{X}$ is $\mathcal{A}$ -theorizable, and let $\mathcal{T}$ denote the associated structural $\mathcal{X}|\mathcal{A}$ -theory $\text{Str}_{\mathcal{A}}(\mathcal{Y})$. Then the following statements hold:

1. $\mathcal{T}$ is λ -extended;
2. If u' admits a left adjoint $f'(A) \in \mathcal{Y}'$ at each $A \in \mathcal{A}'$, then $\mathcal{T}$ is also λ -consistent.

Proof. We begin with 1. Recalling that $\mathcal{T}$ is a full subcategory of $\text{Fun}(\mathcal{Y}, \text{Spc})$, it suffices to show for any $Y \in \mathcal{Y}$ that the composition $\text{ev}_Y \circ \chi : \mathcal{A}^{\text{op}} \rightarrow \text{Spc}$ is left Kan extended from $(\mathcal{A}')^{\text{op}}$, where $\text{ev}_Y : \text{Fun}(\mathcal{Y}, \text{Spc}) \rightarrow \text{Spc}$ denotes evaluation at Y . By definition of χ , this composition identifies with $j_{\mathcal{X}|\mathcal{A}}(u(Y))$. The left Kan extension condition thus holds by the hypothesis on the essential image of u together with the hypothesis that λ is a strict embedding in Prd^{L} .

Now we prove 2. Let $\mathcal{T}'$ be the restriction of $\mathcal{T}$ along λ . Assuming the stated hypothesis on left adjoints, an object $T \in \mathcal{T}'$ is equivalent to $j_{\mathcal{Y}}^{\vee}(f'(A))$ for some $A \in \mathcal{A}'$, and the associated object $\chi_T \in \mathcal{X}$ is then equivalent to $\lambda u' f'(A)$ (see Remark 2.2.12). This verifies the claim. $\square$

Lemma 3.3.18. Let $\lambda : \mathcal{X}'|\mathcal{A}' \rightarrow \mathcal{X}|\mathcal{A}$ be a strict embedding in Prd^{L} and let $\chi : \mathcal{A}^{\text{op}} \rightarrow \mathcal{T}$ be an $\mathcal{X}|\mathcal{A}$ -theory. Let $u : \text{Mod}_{\mathcal{X}}(\mathcal{T}) \rightarrow \mathcal{X}$ be the structure map, $f : \mathcal{X} \rightarrow \text{Mod}_{\mathcal{X}}(\mathcal{T})$ its left adjoint, and $t := uf : \mathcal{X} \rightarrow \mathcal{X}$ the associated monad. Then the following conditions are equivalent:

1. $\mathcal{T}$ is λ -extended;
2. $f|_{\mathcal{A}} : \mathcal{A} \rightarrow \text{Mod}_{\mathcal{X}}(\mathcal{T})$ is right Kan extended along $\lambda : \mathcal{A}' \rightarrow \mathcal{A}$;
3. $t|_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{X}$ is right Kan extended along $\lambda : \mathcal{A}' \rightarrow \mathcal{A}$.

Proof. Consider the commutative diagram

$$\begin{array}{ccc} \mathcal{T}^{\text{op}} & \xrightarrow{j_{\mathcal{T}}^{\vee}} & \text{Mod}_{\mathcal{X}}(\mathcal{T}) \\ \chi^{\text{op}} \uparrow & \nearrow f|_{\mathcal{A}} & \downarrow u \\ \mathcal{A} & \xrightarrow{t|_{\mathcal{A}}} & \mathcal{X} \end{array}$$

(the commutativity of the lower triangle being evident and that of the upper triangle following from Remark 2.2.7). The claim follows from the fact that each of the functors $j_{\mathcal{T}}^{\vee}$ and u preserves and reflects limits. $\square$

Proof of Proposition 3.3.16. Lemma 3.3.17 implies that the right pushforward of an $\mathcal{X}'|\mathcal{A}'$ -theory is λ -consistent and λ -extended. We then get the adjunction of 2 from the following chain of natural equivalences

for $\mathcal{T} \in \text{Thy}_{\mathcal{X}|\mathcal{A}}$ and $\mathcal{T}' \in \text{Thy}_{\mathcal{X}'|\mathcal{A}'}$:

$$\begin{aligned} \text{Map}_{\text{Thy}_{\mathcal{X}|\mathcal{A}}}(\mathcal{T}, \lambda_{**}(\mathcal{T}')) &\simeq \text{Map}_{\text{Cat}_{/\mathcal{X}}}(\lambda_!(\text{Mod}_{\mathcal{X}'}(\mathcal{T}')), \text{Mod}_{\mathcal{X}}(\mathcal{T})) \\ &\simeq \text{Map}_{\text{Cat}_{/\mathcal{X}'}}(\text{Mod}_{\mathcal{X}'}(\mathcal{T}'), \lambda^*(\text{Mod}_{\mathcal{X}}(\mathcal{T}))) \\ &\simeq \text{Map}_{\text{Cat}_{/\mathcal{X}'}}(\text{Mod}_{\mathcal{X}'}(\mathcal{T}'), \text{Mod}_{\mathcal{X}'}(\lambda^*(\mathcal{T}))) \\ &\simeq \text{Map}_{\text{Thy}_{\mathcal{X}'|\mathcal{A}'}}(\lambda^*(\mathcal{T}), \mathcal{T}') \end{aligned}$$

the first from the definition of λ_{**} and the adjunction $\text{Str}_{\mathcal{A}} : \text{Cat}_{/\mathcal{X}} \rightleftarrows \text{PThy}_{\mathcal{A}}^{\text{op}} : \text{Mod}_{\mathcal{X}}$, the second from the adjunction $\lambda_! : \text{Cat}_{/\mathcal{X}'} \rightleftarrows \text{Cat}_{/\mathcal{X}} : \lambda^*$, the third from Proposition 3.3.11, and the fourth from the full faithfulness of $\text{Mod}_{\mathcal{X}'} : \text{Thy}_{\mathcal{X}'|\mathcal{A}'} \rightarrow \text{Cat}_{/\mathcal{X}'}$. Unwinding definitions shows that the associated counit map $\lambda^* \lambda_{**} \rightarrow \text{id}_{\text{Thy}_{\mathcal{X}'|\mathcal{A}'}}$ is the identification of Proposition 3.3.9, giving the full faithfulness assertion of 1.

To finish, it suffices to show that a map between two $\mathcal{X}|\mathcal{A}$ -theories that are each λ -consistent and λ -extended is an equivalence if and only if its restriction along λ is an equivalence. This follows from the full faithfulness of $\text{Mod}_{\mathcal{X}} : \text{Thy}_{\mathcal{X}|\mathcal{A}} \rightarrow \text{Cat}_{/\mathcal{X}}$ together with the combination of Propositions 2.2.26 and 3.3.11 and Lemma 3.3.18. $\square$

Remark 3.3.19. Let $\lambda : \mathcal{X}'|\mathcal{A}' \rightarrow \mathcal{X}|\mathcal{A}$ be a strict embedding in Prd^{L} . Suppose that we are in the situation of Corollary 2.2.30 and that the following conditions also hold:

- the $\mathcal{X}|\mathcal{A}$ -theory $\text{Str}_{\mathcal{A}}(\mathcal{Z})$ is λ -extended (Definition 3.3.15);
- the essential image of u is contained in the essential image of λ , and hence the $\mathcal{X}|\mathcal{A}$ -theory $\text{Str}_{\mathcal{A}}(\mathcal{Y})$ is λ -extended (Lemma 3.3.17).

Then the criterion of Corollary 2.2.30.2 may be simplified: namely, it suffices for the unit map $h(A) \rightarrow \Gamma_v j_v^{\vee}(h(A))$ to be an equivalence for A in the essential image of $\lambda : \mathcal{A}' \rightarrow \mathcal{A}$, as both source and target are right Kan extended along this functor (see Lemma 3.3.18).

§3.4. Derived functors. We return to working with a fixed presented ∞ -category $\mathcal{X}|\mathcal{A}$ in this section. In §2.2, we formulated the notions of $\mathcal{A}$ -theorizability (Definition 2.2.15) and $\mathcal{A}$ -theorization (Definition 2.2.24 and Remark 2.2.25) for ∞ -categories over $\mathcal{X}$. Later in the paper, we'll identify various derived ∞ -categories as instances of the latter. The natural notion of derived functor in this setting can be formulated in full generality, as follows.

Definition 3.4.1. Let $\mathcal{Y}$ be an $\mathcal{A}$ -theorizable ∞ -category over $\mathcal{X}$. Let $\mathcal{T}$ denote the structural $\mathcal{X}|\mathcal{A}$ -theory $\text{Str}_{\mathcal{A}}(\mathcal{Y}) \subseteq \text{Fun}(\mathcal{Y}, \text{Spc})$, so that $\mathcal{Y}$ has $\mathcal{A}$ -theorization $\mathcal{Y}^{\mathcal{A}\text{-th}/\mathcal{X}} = \text{Mod}_{\mathcal{X}}(\mathcal{T})$. We say that a functor $v : \mathcal{Y} \rightarrow \mathcal{Z}$ is $\mathcal{X}|\mathcal{A}$ -derivable if the following conditions hold:

1. at each $T \in \mathcal{T}$, the restricted Yoneda functor $j_v : \mathcal{Z} \rightarrow \text{Fun}(\mathcal{Y}, \text{Spc})^{\text{op}}$ admits a right adjoint $\Gamma_v(T) \in \mathcal{Z}$;
2. the functor $\Gamma_v : \mathcal{T}^{\text{op}} \rightarrow \mathcal{Z}$ given by 1 admits a left Kan extension along the Yoneda embedding $j_{\mathcal{T}}^{\vee} : \mathcal{T}^{\text{op}} \rightarrow \text{Mod}_{\mathcal{X}}(\mathcal{T})$.

When this is the case, we denote the left Kan extension of 2 by

$$v_{\dagger} : \mathcal{Y}^{\mathcal{A}\text{-th}/\mathcal{X}} \rightarrow \mathcal{Z}$$

and refer to this as the $\mathcal{X}|\mathcal{A}$ -derived functor of v .

Example 3.4.2. Let $\mathcal{Y}$ be an $\mathcal{A}$ -small ∞ -category over $\mathcal{X}$ (Definition 2.2.15) and let $\mathcal{Z}$ be an ∞ -category that admits limits and colimits. Then any functor $v : \mathcal{Y} \rightarrow \mathcal{Z}$ is $\mathcal{X}|\mathcal{A}$ -derivable: condition 1 of Definition 3.4.1 holds by the hypotheses that $\text{Str}_{\mathcal{A}}(\mathcal{Y}) \subseteq \mathcal{P}^{\vee}(\mathcal{Y})$ and that $\mathcal{Z}$ admits limits, and condition 2 holds by the hypothesis that $\mathcal{Z}$ admits colimits.

As another example, the universal property of $\mathcal{A}$ -theorization can be understood in these terms.

Proposition 3.4.3. Let $\mathcal{Y}, \mathcal{Z} \in \text{Cat}_{/\mathcal{X}}$ and let $v : \mathcal{Y} \rightarrow \mathcal{Z}$ be a functor over $\mathcal{X}$. Assume that $\mathcal{Y}$ is $\mathcal{A}$ -theorizable and $\mathcal{Z}$ is $\mathcal{A}$ -theoretical, and let $v' : \mathcal{Y}^{\mathcal{A}\text{-th}/\mathcal{X}} \rightarrow \mathcal{Z}$ be the induced functor from the $\mathcal{A}$ -theorization. Then v is $\mathcal{X}|\mathcal{A}$ -derivable, and the derived functor $v_{\dagger} : \mathcal{Y}^{\mathcal{A}\text{-th}/\mathcal{X}} \rightarrow \mathcal{Z}$ is canonically equivalent to v' .

Proof. Using Remark 3.2.13 to identify the theorization over $\mathcal{X}$ under question with a theorization over $\mathcal{Z}$, we may assume without loss of generality that $\mathcal{Z} = \mathcal{X}$. Thus v identifies with the structure map $u : \mathcal{Y} \rightarrow \mathcal{X}$ and v' with the structure map $u' : \mathcal{Y}^{\mathcal{A}\text{-th}}/\mathcal{X} \rightarrow \mathcal{X}$.

Let $\mathcal{T} := \text{Str}_{\mathcal{A}}(\mathcal{Y})$. Proposition 2.2.27 says that $u' \circ j_{\mathcal{T}}^{\vee}$ is a partial right adjoint of $j_u : \mathcal{X} \rightarrow \text{Fun}(\mathcal{Y}, \text{Spc})^{\text{op}}$, and Proposition 2.2.8.2 says that u' is left Kan extended along $j_{\mathcal{T}}^{\vee}$. This proves the claim. $\square$

Ch. 4. Spectral algebra

In Ch. 2, we discussed algebraic structure relative to a general presentable ∞ -category $\mathcal{X}$ (most of the discussion depending also on a choice of presentation of $\mathcal{X}$). The simplest examples of the picture developed there, covering the cases $\mathcal{X} = \mathbf{Spc}$ and $\mathcal{X} = \mathbf{Set}$, were explained in Remarks 2.3.13 and 2.3.15. This chapter is devoted to elaborating a slightly more complicated set of examples, covering in particular the case $\mathcal{X} = \mathbf{Spt}$, the ∞ -category of spectra. So, essentially, our subject here is algebraic structure on spectra. We'll work with a more abstract setup, however.

Notation 4.0.1. Throughout this section, we let $\mathcal{X}$ be a stable, presentable ∞ -category and we let $\mathcal{A} \subseteq \mathcal{X}$ be an essentially small, detecting subcategory (Convention 0.3.11) of compact objects that contains the zero object and is closed under desuspensions.

Example 4.0.2. We may take $\mathcal{X} = \mathbf{Spt}$ and $\mathcal{A}$ to be either of the following subcategories thereof:

- the full subcategory $\mathcal{A}$ generated under finite sums and desuspensions by $\mathbb{S}$;
- the full subcategory $\mathcal{A}^\circ$ spanned by 0 and $\mathbb{S}[-n]$ for $n \in \mathbb{Z}_{\geq 0}$.

More generally, for any set I , we may take $\mathcal{X} = \mathbf{Spt}^I$ and $\mathcal{A}$ to be either of the following:

- the full subcategory $\mathcal{A}^I$ generated under finite sums and desuspensions by $\mathbb{S}(-i)$ for $i \in I$ (see Convention 0.3.12);
- the full subcategory $\mathcal{A}_I^\circ$ spanned by 0 and $\mathbb{S}(-i)[-n]$ for $i \in I$ and $n \in \mathbb{Z}_{\geq 0}$.

Remark 4.0.3. The pairs of subcategories in Example 4.0.2 fit the following general pattern. Suppose that $\mathcal{A}$ is closed under finite sums in $\mathcal{X}$ and let $\mathcal{A}^\circ \subseteq \mathcal{A}$ be a full subcategory that generates $\mathcal{A}$ under finite sums. Then $\mathcal{A}^\circ$ is also a detecting subcategory of $\mathcal{X}$; thus, if it contains 0 and is closed under desuspensions, it also satisfies the hypotheses of Notation 4.0.1.

In §4.1 and §4.2, we'll show that $\mathcal{A}$ defines a presentation of $\mathcal{X}$ and we'll put the key definitions and results of Ch. 2 in a more explicit form in this context. The subsequent §4.3 and §4.4 deal with a couple of additional matters specific to this context: the former characterizes those algebraic structures over $\mathcal{X}$ that again comprise stable ∞ -categories; and the latter deals with compatibility between algebraic structure over $\mathcal{X}$ and t-structures on $\mathcal{X}$ (in simple cases).

Remark 4.0.4. The main ideas of this chapter overlap with previous work of Balderrama, Hopkins–Lurie, and Pstrągowski. In particular, the type of algebraic theories and models that we'll consider may be seen as variants of the notions of *loop theory* and *loop model*, which were studied in special cases by Hopkins–Lurie [20] and Pstrągowski [32], then formulated in general by Balderrama [6], and then further developed in joint work of Balderrama–Pstrągowski [7]. With respect to this comparison, the essential new point here is the consideration of theorization in this context (which is the topic of §4.2). Another point worth emphasizing is the preference of desuspensions in our choice of presentation of $\mathcal{X}$, which is important for our applications to derived ∞ -categories later in the paper; as a result of this, our notions in fact compare more directly to the notions of *deloop theory* and *deloop model* in [7].

Another form of these ideas may be found in work of Brantner–Mathew [12]; see Remark 4.1.16 for a more detailed comment on this, as well as Ch. 5, where we will return to their work in more depth.

§4.1. Theories.

Definition 4.1.1. Given a functor $f : \mathcal{A} \rightarrow \mathcal{E}$ that preserves the initial object, we say that f *preserves suspensions* if, for each $A \in \mathcal{A}$, it carries the suspension diagram

$$\begin{array}{ccc} A[-1] & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & A \end{array}$$

in $\mathcal{A}$ to a cocartesian square in $\mathcal{E}$. Dually, given a functor $f : \mathcal{A}^{\text{op}} \rightarrow \mathcal{E}$ that preserves the final object, we

say that f *preserves loopings* if the opposite functor $f^{\text{op}} : \mathcal{A} \rightarrow \mathcal{E}^{\text{op}}$ preserves suspensions (in other words, f carries the above suspension diagrams to cartesian squares in $\mathcal{E}$).

Theorem 4.1.2. *The full subcategory $\mathcal{A} \subseteq \mathcal{X}$ is dense, and the essential image of the embedding $j_{\mathcal{X}|\mathcal{A}} : \mathcal{X} \rightarrow \mathcal{P}(\mathcal{A})$ may be characterized in both of the following ways:*

1. *It consists of those presheaves $\chi : \mathcal{A}^{\text{op}} \rightarrow \mathbf{Spc}$ that preserve the final object and loopings.*
2. *Suppose that $\mathcal{A}$ is closed under finite sums in $\mathcal{X}$ and let $\mathcal{A}^\circ \subseteq \mathcal{A}$ be any full subcategory that contains 0, is closed under desuspensions, and generates $\mathcal{A}$ under finite sums. Then the essential image consists of those presheaves $\chi : \mathcal{A}^{\text{op}} \rightarrow \mathbf{Spc}$ that preserve finite products and whose restriction to $(\mathcal{A}^\circ)^{\text{op}}$ preserves loopings.*

A more general form of Theorem 4.1.2.2 may be found in the work of Balderamma–Pstrągowski [7, Proposition 7.2.3], the proof of which references a similar result from the earlier work of Balderamma [6, Theorem 7.3.3]. We'll give a proof of Theorem 4.1.2 at the end of this section, currently moving on to consider $\mathcal{X}|\mathcal{A}$ -theories and their models in $\mathcal{X}$ (Definitions 2.2.1 and 2.2.3). Let us begin by unravelling the meaning of the former notion a little bit.

Definition 4.1.3. Let $\mathcal{E}$ be an ∞ -category admitting a final object $1 \in \mathcal{E}$, and let $\mathcal{E}_* := \mathcal{E}_{1/}$ be the ∞ -category of pointed objects in $\mathcal{E}$. By a *spectrum object* X of $\mathcal{E}$ we mean a sequence $\{\Omega^{\infty-n} X\}_{n \in \mathbb{Z}_{\geq 0}}$ of objects of $\mathcal{E}_*$, together with pullback diagrams

$$\begin{array}{ccc} \Omega^{\infty-n} X & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & \Omega^{\infty-n-1} X \end{array}$$

in $\mathcal{E}_*$ for each $n \in \mathbb{Z}_{\geq 0}$; we refer to $\Omega^{\infty-n-1} X$ as a *delooping* of $\Omega^{\infty-n} X$.

Remark 4.1.4. Let $\mathcal{T}$ be a locally small ∞ -category and let $\chi : \mathcal{A}^{\text{op}} \rightarrow \mathcal{T}$ be a functor (resp. essentially surjective functor, i.e. $\mathcal{A}$ -theory). By Theorem 4.1.2 and Remark 2.1.6, χ is an $\mathcal{X}|\mathcal{A}$ -object (resp. $\mathcal{X}|\mathcal{A}$ -theory) if and only if it preserves the final object and loopings; also, supposing that $\mathcal{A}$ is closed under finite sums in $\mathcal{X}$ and that $\mathcal{A}^\circ \subseteq \mathcal{A}$ is a full subcategory that contains 0, is closed under desuspensions, and generates $\mathcal{A}$ under finite sums, these conditions are equivalent to χ preserving finite products and its restriction to $(\mathcal{A}^\circ)^{\text{op}}$ preserving loopings.

If χ is an $\mathcal{X}|\mathcal{A}$ -object, then for each $A \in \mathcal{A}$, the object $\chi(A)$ canonically extends to a spectrum object of $\mathcal{T}$: each such object has a pointing induced by the unique map $A \rightarrow 0$ in $\mathcal{A}$, and then furthermore has the sequence of deloopings $\{\chi(A[-n])\}_{n \in \mathbb{Z}_{\geq 0}}$.

Example 4.1.5. Let I be a set and suppose that $\mathcal{X} = \mathbf{Spt}^I$ and $\mathcal{A} = \mathbf{A}_I^\circ$ (Example 4.0.2). Let $\mathcal{T}$ be a locally small ∞ -category admitting a final object. By Remark 4.1.4, a $\mathbf{Spt}^I|\mathbf{A}_I^\circ$ -object $\chi : (\mathbf{A}_I^\circ)^{\text{op}} \rightarrow \mathcal{T}$ determines, for each $i \in I$, a spectrum object χ_i of $\mathcal{T}$ with

$$\Omega^{\infty-n} \chi_i = \chi(\mathbb{S}(-i)[-n])$$

for $n \in \mathbb{Z}_{\geq 0}$. This construction gives an equivalence between $\mathbf{Spt}^I|\mathbf{A}_I^\circ$ -objects of $\mathcal{T}$ and I -indexed collections of spectrum objects of $\mathcal{T}$.

Example 4.1.6. Let I be a set and suppose that $\mathcal{X} = \mathbf{Spt}^I$ and $\mathcal{A} = \mathbf{A}^I$ (Example 4.0.2). It follows from Example 4.1.5 that we have an equivalence between the following two types of data:

1. a $\mathbf{Spt}^I|\mathbf{A}^I$ -theory $\chi : (\mathbf{A}^I)^{\text{op}} \rightarrow \mathcal{T}$;
2. a locally small ∞ -category $\mathcal{T}$ admitting finite products, together with an I -indexed collection $\{\chi_i\}_{i \in I}$ of spectrum objects of $\mathcal{T}$, such that $\mathcal{T}$ is generated under finite products by the objects $\Omega^{\infty-n} \chi_i$ for $i \in I$ and $n \in \mathbb{Z}_{\geq 0}$.

We may alternatively refer to $\mathbf{Spt}^I|\mathbf{A}^I$ -theories as *I-sorted spectral algebraic theories*.

We now turn to models of $\mathcal{X}|\mathcal{A}$ -theories. These can be made more explicit in the same way that

Remark 4.1.4 does for $\mathcal{X}|\mathcal{A}$ -theories themselves. Let us spell this out and then reformulate it in terms of universal properties characterizing the ∞ -categories of such models.

Definition 4.1.7. Let $\chi : \mathcal{A}^{\text{op}} \rightarrow \mathcal{T}$ be an $\mathcal{X}|\mathcal{A}$ -object. Given a functor $f : \mathcal{T} \rightarrow \mathcal{E}$ that preserves the final object, we say that f *preserves χ -loopings* if $f \circ \chi$ preserves loopings. Dually, given a functor $f : \mathcal{T}^{\text{op}} \rightarrow \mathcal{E}$ that preserves the initial object, we say that f *preserves χ -suspensions* if $f \circ \chi^{\text{op}}$ preserves suspensions.

Remark 4.1.8. Let $\chi : \mathcal{A}^{\text{op}} \rightarrow \mathcal{T}$ be an $\mathcal{X}|\mathcal{A}$ -theory. Recall that the models of $\mathcal{T}$ in $\mathcal{X}$ span a full subcategory $\text{Mod}_{\mathcal{X}}(\mathcal{T}) \subseteq \text{Fun}(\mathcal{T}, \text{Spc})$. It follows from Theorem 4.1.2 that a functor $M : \mathcal{T} \rightarrow \text{Spc}$ is a model of $\mathcal{T}$ in $\mathcal{X}$ if and only if it preserves the final object and χ -loopings. Alternatively, supposing that $\mathcal{A}$ is closed under finite sums in $\mathcal{X}$ and $\mathcal{A}^{\circ} \subseteq \mathcal{A}$ is a full subcategory that contains 0, is closed under desuspensions, and generates $\mathcal{A}$ under finite sums, a functor $M : \mathcal{T} \rightarrow \text{Spc}$ is a model of $\mathcal{T}$ in $\mathcal{X}$ if and only if it preserves finite products and preserves $\chi|_{(\mathcal{A}^{\circ})^{\text{op}}}$ -loopings.

Remark 4.1.9. Let $\mathcal{T}$ be an $\mathcal{X}|\mathcal{A}$ -theory. By Remark 4.1.8 and the fact that filtered colimits commute with finite limits in Spc , the inclusion functor $\text{Mod}_{\mathcal{X}}(\mathcal{T}) \hookrightarrow \text{Fun}(\mathcal{T}, \text{Spc})$ and structure map $\text{Mod}_{\mathcal{X}}(\mathcal{T}) \rightarrow \mathcal{X}$ preserve filtered colimits. This is also a special case of Corollary 2.2.9.2, since the objects of $\mathcal{A}$ are compact in $\mathcal{X}$. It follows that $\text{Mod}_{\mathcal{X}}(\mathcal{T})$ is compactly generated.

Remark 4.1.10. Let $\chi : \mathcal{A}^{\text{op}} \rightarrow \mathcal{T}$ be an $\mathcal{X}|\mathcal{A}$ -theory. By Remark 2.2.7, $\text{Mod}_{\mathcal{X}}(\mathcal{T})$ is a Bousfield localization of $\mathcal{P}^{\vee}(\mathcal{T}) = \text{Fun}(\mathcal{T}, \text{Spc})$. This can be understood more explicitly in the present context. Letting $0 \in \mathcal{P}^{\vee}(\mathcal{T})$ denote the initial object, we have a canonical map

$$0 \rightarrow j_{\mathcal{T}}^{\vee}(1)$$

in $\mathcal{P}^{\vee}(\mathcal{T})$; and for each $A \in \mathcal{A}$, we have a canonical map

$$j_{\mathcal{T}}^{\vee}(1) \sqcup_{j_{\mathcal{T}}^{\vee}(\chi(A[-1]))} j_{\mathcal{T}}^{\vee}(1) \rightarrow j_{\mathcal{T}}^{\vee}(\chi(A))$$

in $\mathcal{P}^{\vee}(\mathcal{T})$. Let Q denote the set of these maps. Then Remark 4.1.8 says that $\text{Mod}_{\mathcal{X}}(\mathcal{T})$ is the full subcategory of Q -local objects in $\mathcal{P}^{\vee}(\mathcal{T})$, hence is a Bousfield localization by [28, Tag 06VH].

Proposition 4.1.11. *Let $\chi : \mathcal{A}^{\text{op}} \rightarrow \mathcal{T}$ be an $\mathcal{X}|\mathcal{A}$ -theory and let $\mathcal{E}$ be a cocomplete ∞ -category. Let $\text{Fun}'(\text{Mod}_{\mathcal{X}}(\mathcal{T}), \mathcal{E}) \subseteq \text{Fun}(\text{Mod}_{\mathcal{X}}(\mathcal{T}), \mathcal{E})$ be the full subcategory spanned by those functors that preserve colimits. Then the following statements hold:*

1. *Let $\text{Fun}'_1(\mathcal{T}^{\text{op}}, \mathcal{E}) \subseteq \text{Fun}(\mathcal{T}^{\text{op}}, \mathcal{E})$ be the full subcategory spanned by those functors $f : \mathcal{T}^{\text{op}} \rightarrow \mathcal{E}$ that preserve the initial object and χ -suspensions. Then restriction and left Kan extension along the Yoneda embedding $j_{\mathcal{T}}^{\vee} : \mathcal{T}^{\text{op}} \hookrightarrow \text{Mod}_{\mathcal{X}}(\mathcal{T})$ define an equivalence*

$$\text{Fun}'(\text{Mod}_{\mathcal{X}}(\mathcal{T}), \mathcal{E}) \simeq \text{Fun}'_1(\mathcal{T}^{\text{op}}, \mathcal{E}).$$

2. *Assume that $\mathcal{A}$ is closed under finite sums in $\mathcal{X}$ and let $\mathcal{A}^{\circ} \subseteq \mathcal{A}$ be a full subcategory that contains 0, is closed under desuspensions, and generates $\mathcal{A}$ under finite sums. Let $\text{Fun}'_2(\mathcal{T}^{\text{op}}, \mathcal{E}) \subseteq \text{Fun}(\mathcal{T}^{\text{op}}, \mathcal{E})$ be the full subcategory spanned by those functors $f : \mathcal{T}^{\text{op}} \rightarrow \mathcal{E}$ that preserve finite coproducts and $\chi|_{(\mathcal{A}^{\circ})^{\text{op}}}$ -suspensions. Then restriction and left Kan extension along the Yoneda embedding $j_{\mathcal{T}}^{\vee} : \mathcal{T}^{\text{op}} \hookrightarrow \text{Mod}_{\mathcal{X}}(\mathcal{T})$ define an equivalence*

$$\text{Fun}'(\text{Mod}_{\mathcal{X}}(\mathcal{T}), \mathcal{E}) \simeq \text{Fun}'_2(\mathcal{T}^{\text{op}}, \mathcal{E}).$$

Proof. We'll just address 1; a similar argument proves 2. Let $\text{Fun}'(\mathcal{P}^{\vee}(\mathcal{T}), \mathcal{E}) \subseteq \text{Fun}(\mathcal{P}^{\vee}(\mathcal{T}), \mathcal{E})$ be the full subcategory spanned by those functors that preserve colimits, and recall that restriction and left Kan extension along $j_{\mathcal{T}}^{\vee} : \mathcal{T}^{\text{op}} \rightarrow \mathcal{P}^{\vee}(\mathcal{T})$ define an equivalence $\text{Fun}'(\mathcal{P}^{\vee}(\mathcal{T}), \mathcal{E}) \simeq \text{Fun}(\mathcal{T}^{\text{op}}, \mathcal{E})$. The inclusion functor $\text{Mod}_{\mathcal{X}}(\mathcal{T}) \hookrightarrow \mathcal{P}^{\vee}(\mathcal{T})$ admits a left adjoint $L : \mathcal{P}^{\vee}(\mathcal{T}) \rightarrow \text{Mod}_{\mathcal{X}}(\mathcal{T})$ (Remark 4.1.10), and a functor $f : \text{Mod}_{\mathcal{X}}(\mathcal{T}) \rightarrow \mathcal{E}$ preserves colimits if and only if the composition $f \circ L : \mathcal{P}^{\vee}(\mathcal{T}) \rightarrow \mathcal{E}$ preserves colimits. Thus, to establish the claim, it suffices to show that the restriction functor $L^* : \text{Fun}'(\text{Mod}_{\mathcal{X}}(\mathcal{T}), \mathcal{E}) \rightarrow \text{Fun}'(\mathcal{P}^{\vee}(\mathcal{T}), \mathcal{E})$ is fully faithful and has essential image comprised of those functors $g : \mathcal{P}^{\vee}(\mathcal{T}) \rightarrow \mathcal{E}$ whose restriction to $\mathcal{T}^{\text{op}}$ lies in $\text{Fun}'_1(\mathcal{T}^{\text{op}}, \mathcal{E})$.

Let Q be the set of maps in $\mathcal{P}^\vee(\mathcal{T})$ defined in Remark 4.1.10, so that $\text{Mod}_{\mathcal{X}}(\mathcal{T})$ is the full subcategory of Q -local objects in $\mathcal{P}^\vee(\mathcal{T})$. Let $\overline{Q}$ denote the strongly saturated class of maps in $\mathcal{P}^\vee(\mathcal{T})$ generated by Q [27, Definition 5.5.4.5 and Remark 5.5.4.7]. By [27, Propositions 5.2.7.12 and 5.5.4.15], the functor $L^* : \text{Fun}(\text{Mod}_{\mathcal{X}}(\mathcal{T}), \mathcal{E}) \rightarrow \text{Fun}(\mathcal{P}^\vee(\mathcal{T}), \mathcal{E})$ is fully faithful and has essential image consisting of those functors g that invert the elements of $\overline{Q}$. To complete the argument, we must show that a colimit preserving functor $g : \mathcal{P}^\vee(\mathcal{T}) \rightarrow \mathcal{E}$ inverts the elements of $\overline{Q}$ if and only if its restriction to $\mathcal{T}^{\text{op}}$ preserves finite coproducts and its further restriction to $\mathcal{A}$ along χ^{op} preserves suspensions. The latter condition is equivalent to g inverting the elements of Q , so this follows from the fact that the collection of maps inverted by a colimit preserving functor g is strongly saturated. $\square$

There is also a variant of Proposition 4.1.11 that characterizes sifted colimit preserving functors out of the ∞ -category of models of an $\mathcal{X}|\mathcal{A}$ -theory, when $\mathcal{A}$ is closed under finite sums in $\mathcal{X}$. For this we recall that loops and suspensions can also be described via cobar and bar constructions.

Remark 4.1.12. Let $\mathcal{E}$ be an ∞ -category. If $\mathcal{E}$ admits finite coproducts, then for any maps $B \leftarrow A \rightarrow C$ in $\mathcal{E}$, there is a canonical simplicial object $\text{bar}_A(B, C)_\bullet : \Delta^{\text{op}} \rightarrow \mathcal{E}$, given on objects by the formula $\text{bar}_A(B, C)_n \simeq B \sqcup A^{\sqcup n} \sqcup C$, such that a geometric realization of $\text{bar}_A(B, C)_\bullet$ is equivalent to a pushout $B \sqcup_A C$ [5, Lemma 2.7].

If $\mathcal{E}$ admits finite products, then carrying out the above in $\mathcal{E}^{\text{op}}$ gives the following: for any maps $B \rightarrow A \leftarrow C$ in $\mathcal{E}$, there is a canonical cosimplicial object $\text{cobar}_A(B, C)^\bullet : \Delta \rightarrow \mathcal{E}$, given on objects by the formula $\text{cobar}_A(B, C)_n \simeq B \times A^{\times n} \times C$, such that a totalization of $\text{cobar}_A(B, C)^\bullet$ is equivalent to a pullback $B \times_A C$.

Definition 4.1.13. Assume that $\mathcal{A}$ is closed under finite sums in $\mathcal{X}$. We say that a functor $f : \mathcal{A} \rightarrow \mathcal{E}$ *preserves bar constructions* if, for each $A, B \in \mathcal{A}$, it preserves the geometric realization

$$\text{colim}_{\Delta^{\text{op}}} (B \oplus \text{bar}_{A[-1]}(0, 0)_\bullet) \simeq B \oplus A.$$

Dually, we say that a functor $f : \mathcal{A}^{\text{op}} \rightarrow \mathcal{E}$ *preserves cobar constructions* if $f^{\text{op}} : \mathcal{A} \rightarrow \mathcal{E}^{\text{op}}$ preserves bar constructions.

For $\chi : \mathcal{A}^{\text{op}} \rightarrow \mathcal{T}$ an $\mathcal{X}|\mathcal{A}$ -object, we say that a functor $f : \mathcal{T} \rightarrow \mathcal{E}$ *preserves χ -cobar constructions* if $f \circ \chi$ preserves cobar constructions, and we say that a functor $f : \mathcal{T}^{\text{op}} \rightarrow \mathcal{E}$ *preserves χ -bar constructions* if $f \circ \chi^{\text{op}}$ preserves bar constructions.

Remark 4.1.14. Regarding the conditions of Definition 4.1.13:

1. Let $\mathcal{A}^\circ \subseteq \mathcal{A}$ be a full subcategory that generates $\mathcal{A}$ under finite sums. Then it suffices to check that the stated geometric realization is preserved for $A \in \mathcal{A}^\circ$ (but still $B \in \mathcal{A}$).
2. A finite coproduct preserving functor $f : \mathcal{A} \rightarrow \mathcal{E}$ preserves bar constructions if and only if it preserves suspensions, and dually a finite product preserving functor $f : \mathcal{A}^{\text{op}} \rightarrow \mathcal{E}$ preserves cobar constructions if and only if it preserves loopings.

Proposition 4.1.15. Assume that $\mathcal{A}$ is closed under finite sums in $\mathcal{X}$. Let $\chi : \mathcal{A}^{\text{op}} \rightarrow \mathcal{T}$ be an $\mathcal{X}|\mathcal{A}$ -theory and let $\mathcal{E}$ be an ∞ -category that admits sifted colimits. Let $\text{Fun}''(\text{Mod}_{\mathcal{X}}(\mathcal{T}), \mathcal{E}) \subseteq \text{Fun}(\text{Mod}_{\mathcal{X}}(\mathcal{T}), \mathcal{E})$ be the full subcategory spanned by those functors that preserve sifted colimits, and let $\text{Fun}''(\mathcal{T}^{\text{op}}, \mathcal{E}) \subseteq \text{Fun}(\mathcal{T}^{\text{op}}, \mathcal{E})$ be the full subcategory spanned by those functors $f : \mathcal{T}^{\text{op}} \rightarrow \mathcal{E}$ such that preserve χ -bar constructions. Then restriction and left Kan extension along the Yoneda embedding $j_{\mathcal{T}}^\vee : \mathcal{T}^{\text{op}} \hookrightarrow \text{Mod}_{\mathcal{X}}(\mathcal{T})$ define an equivalence

$$\text{Fun}''(\text{Mod}_{\mathcal{X}}(\mathcal{T}), \mathcal{E}) \simeq \text{Fun}''(\mathcal{T}^{\text{op}}, \mathcal{E}).$$

Proof. This may be proved similarly to Proposition 4.1.11; let us just indicate what modifications to the argument there are necessary here. First, instead of $\mathcal{P}^\vee(\mathcal{T})$, we use the full subcategory $\mathcal{P}_\Sigma^\vee(\mathcal{T})$ thereof spanned by the finite product preserving functors; recall that, letting $\text{Fun}''(\mathcal{P}_\Sigma^\vee(\mathcal{T}), \mathcal{E}) \subseteq \text{Fun}(\mathcal{P}_\Sigma^\vee(\mathcal{T}), \mathcal{E})$ be the full subcategory spanned by those functors that preserve sifted colimits, restriction and left Kan extension along $j_{\mathcal{T}}^\vee : \mathcal{T}^{\text{op}} \rightarrow \mathcal{P}_\Sigma^\vee(\mathcal{T})$ define an equivalence $\text{Fun}''(\mathcal{P}_\Sigma^\vee(\mathcal{T}), \mathcal{E}) \simeq \text{Fun}(\mathcal{T}, \mathcal{E})$ [27, Proposition 5.5.8.15]. Second, we replace the set Q of maps in $\mathcal{P}^\vee(\mathcal{T})$ with the set R of maps in $\mathcal{P}_\Sigma^\vee(\mathcal{T})$ consisting of

the canonical maps

$$\operatorname{colim}_{\Delta_{\text{op}}^{\vee}} j_{\mathcal{T}}^{\vee}(T \times \operatorname{cobar}_{\chi(A[-1])}(1, 1)^{\bullet}) \rightarrow j_{\mathcal{T}}^{\vee}(T \times \chi(A)),$$

or equivalently

$$j_{\mathcal{T}}^{\vee}(T) \sqcup \operatorname{colim}_{\Delta_{\text{op}}^{\vee}} \operatorname{bar}_{j_{\mathcal{T}}^{\vee}(\chi(A[-1]))}(j_{\mathcal{T}}^{\vee}(1), j_{\mathcal{T}}^{\vee}(1))^{\bullet} \rightarrow j_{\mathcal{T}}^{\vee}(T) \sqcup j_{\mathcal{T}}^{\vee}(\chi(A)),$$

for $A \in \mathcal{A}$ and $T \in \mathcal{T}$. Finally, one additional observation is needed in the last part of the argument: namely, that the strongly saturated class $\overline{R}$ generated by R is the smallest class containing R that is closed under sifted colimits and the two out of three property. This follows from [5, Lemma 2.10], since R is by definition closed under taking coproducts with identity maps in $\mathcal{T}$; and it implies that any sifted colimit preserving functor $g : \mathcal{P}_{\Sigma}^{\vee}(\mathcal{T}) \rightarrow \mathcal{E}$ that inverts R in fact inverts $\overline{R}$. $\square$

Remark 4.1.16. The geometric realizations appearing in Definition 4.1.13 are examples of *finite stable geometric realizations* in the sense of Brantner–Campos–Nuiten [10, Notation 2.36], a generalization of the *finite coconnective geometric realizations* featuring in the work of Brantner–Mathew [12, Definition 3.10]. Thus, in the special case that $\mathcal{T} = \mathcal{A}^{\text{op}}$, so that $\operatorname{Mod}_{\mathcal{X}}(\mathcal{T}) \simeq \mathcal{X}$, Proposition 4.1.15 may be seen as a variant of [12, Proposition 3.14] and [10, Proposition 2.40].

To complete this section, let us return and prove Theorem 4.1.2. We'll first address the basic case $\mathcal{X} = \operatorname{Spt}^I$ and then deduce the general statement.

Lemma 4.1.17. *Let I be a set. Then the full subcategory $A_I^{\circ} \subseteq \operatorname{Spt}^I$ (Example 4.0.2) is dense, and the essential image of the restricted Yoneda functor $j_{\operatorname{Spt}^I|A_I^{\circ}} : \operatorname{Spt}^I \rightarrow \mathcal{P}(A_I^{\circ})$ consists of those presheaves $\chi : (A_I^{\circ})^{\text{op}} \rightarrow \operatorname{Spc}$ that preserve the final object and loopings.*

Proof. Let $\mathcal{Q}$ be the ∞ -category given by the nerve of $\mathbb{Z}_{\geq -1} \times \mathbb{Z}_{\geq -1}$, where $\mathbb{Z}_{\geq -1}$ is equipped with the partial order $\geq$ and $\mathbb{Z}_{\geq -1} \times \mathbb{Z}_{\geq -1}$ with the product partial order. Let $\mathcal{M} \subseteq \mathcal{P}(I \times \mathcal{Q})$ be the full subcategory spanned by those presheaves $M : (I \times \mathcal{Q})^{\text{op}} \rightarrow \operatorname{Spc}$ for which the following conditions hold:

- a. $M(i, m, n)$ is contractible for all $i \in I$ and $m, n \in \mathbb{Z}_{\geq -1}$ such that either $m \neq n$ or $m = n = -1$;
- b. the induced commutative square of spaces

$$\begin{array}{ccc} M(i, n, n) & \longrightarrow & M(i, n+1, n) \\ \downarrow & & \downarrow \\ M(i, n, n+1) & \longrightarrow & M(i, n+1, n+1) \end{array}$$

is cartesian for all $i \in I$ and $n \in \mathbb{Z}_{\geq 0}$.

For $M \in \mathcal{M}$, the unique map $(n, n) \rightarrow (-1, -1)$ in $\mathcal{Q}$ determines a pointing of the space $M(i, n, n)$ for each $i \in I$ and $n \in \mathbb{Z}_{\geq 0}$, and the above conditions imply that we have a canonical equivalence of pointed spaces $\Omega M(i, n+1, n+1) \simeq M(i, n, n)$ for each $i \in I$ and $n \in \mathbb{Z}_{\geq 0}$. In this way, we obtain a functor $\alpha : \mathcal{M} \rightarrow \operatorname{Spt}^I$. An argument similar to that in [24, Proof of Proposition 8.14] shows that this functor is an equivalence.

Similar to Remark 4.1.10, $\mathcal{M}$ is an accessible localization of $\mathcal{P}(I \times \mathcal{Q})$; let $L : \mathcal{P}(I \times \mathcal{Q}) \rightarrow \mathcal{M}$ be the localization functor. Let $q : I \times \mathcal{Q} \rightarrow \operatorname{Spt}^I$ be the composition

$$I \times \mathcal{Q} \xrightarrow{j_{I \times \mathcal{Q}}} \mathcal{P}(I \times \mathcal{Q}) \xrightarrow{L} \mathcal{M} \xrightarrow{\alpha} \operatorname{Spt}^I.$$

The previous paragraph tells us that q is dense and that the essential image of $j_{\operatorname{Spt}^I|I \times \mathcal{Q}}$ is $\mathcal{M}$. Letting $\mathcal{A}^{\circ} \subseteq \operatorname{Spt}^I$ denote the essential image of q , Proposition 2.1.11 then tells us that $\mathcal{A}^{\circ}$ is a dense subcategory of Spt^I and that a presheaf $\chi : (\mathcal{A}^{\circ})^{\text{op}} \rightarrow \operatorname{Spc}$ is in the essential image of $j_{\operatorname{Spt}^I|\mathcal{A}^{\circ}}$ if and only if $\chi \circ q \in \mathcal{M}$. Unwinding the definitions shows that $\mathcal{A}^{\circ} = A_I^{\circ}$ and that $\chi \circ q$ satisfies a and b if and only if χ preserves the final object and loopings. $\square$

Proof of Theorem 4.1.2. We begin with 1. In light of Lemma 4.1.17 and Theorem 2.3.10, it suffices to exhibit, for some set I , a strongly algebraic functor $u : \mathcal{X} \rightarrow \operatorname{Spt}^I$ such that the essential image of A_I° under the left adjoint $f : \operatorname{Spt}^I \rightarrow \mathcal{X}$ of u is equal to $\mathcal{A}$. This may be obtained by choosing a set of objects $\{A_i\}_{i \in I}$

in $\mathcal{A}$ representing every equivalence class, and then defining $u : \mathcal{X} \rightarrow \mathbf{Spt}^I$ to be the functor corepresented by this set of objects, i.e. setting $u_i := \underline{\mathrm{Map}}_{\mathcal{X}}(A_i, -)$; the hypothesis that $\mathcal{A}$ is a detecting subcategory of compact objects implies that this functor u is conservative and colimit preserving, in addition to being limit preserving for general reasons.

For 2, note that we may apply 1 with $\mathcal{A}^\circ$ in place of $\mathcal{A}$. The claim follows from this and Remark 2.3.8. $\square$

§4.2. Theorization. In the previous section, we discussed what $\mathcal{X}|\mathcal{A}$ -theories and their models look like for $\mathcal{X}$ and $\mathcal{A}$ as in Notation 4.0.1. Here we'll continue reviewing the main points from Ch. 2 in this context, beginning with those remaining from §2.2.

Remark 4.2.1. To $\mathcal{Y} \in \mathrm{Cat}_{/\mathcal{X}}$ we have associated a full subcategory $\mathrm{Str}_{\mathcal{A}}(\mathcal{Y}) \subseteq \mathrm{Fun}(\mathcal{Y}, \mathrm{Spc})$: it is defined by an essentially surjective functor $\chi : \mathcal{A}^{\mathrm{op}} \rightarrow \mathrm{Str}_{\mathcal{A}}(\mathcal{Y})$, given by the formula $\chi(A)(Y) \simeq \mathrm{Map}_{\mathcal{X}}(A, u(Y))$, where $u : \mathcal{Y} \rightarrow \mathcal{X}$ is the structure map (Construction 2.2.11). We say that $\mathcal{Y}$ is $\mathcal{A}$ -theorizable when $\mathrm{Str}_{\mathcal{A}}(\mathcal{Y})$ is locally small. When this holds, $\mathrm{Str}_{\mathcal{A}}(\mathcal{Y})$ (or χ) is an $\mathcal{X}|\mathcal{A}$ -theory: this is generally true by Lemma 2.2.21, and it's also clear in the present context from the characterization of $\mathcal{X}|\mathcal{A}$ -theories in Remark 4.1.4.

Example 4.2.2. Suppose that $\mathcal{X} = \mathbf{Spt}^I$ and $\mathcal{A} = \mathbf{A}^I$ for some set I (Example 4.0.2). For $\mathcal{Y} \in \mathrm{Cat}_{/\mathbf{Spt}^I}$, the full subcategory $\mathrm{Str}_{\mathbf{A}^I}(\mathcal{Y}) \subseteq \mathrm{Fun}(\mathcal{Y}, \mathrm{Spc})$ is the one generated under finite products by the functors $\Omega^{\infty-n}u_i : \mathcal{Y} \rightarrow \mathrm{Spc}$ for $i \in I$ and $n \in \mathbb{Z}_{\geq 0}$, and the structure map $\chi : (\mathbf{A}^I)^{\mathrm{op}} \rightarrow \mathrm{Str}_{\mathbf{A}^I}(\mathcal{Y})$ carries $\mathbb{S}(-i)[-n]$ to $\Omega^{\infty-n}u_i$. That this is a $\mathbf{Spt}^I|\mathbf{A}^I$ -theory (a.k.a. I -sorted spectral algebraic theory) when it is locally small is clear from the concrete characterization of such theories given in Example 4.1.6: the corresponding I -indexed collection of spectrum objects $\{\chi_i\}_{i \in I}$ is simply the collection of component functors $\{u_i\}_{i \in I}$ (under the canonical identification between spectrum objects of $\mathrm{Fun}(\mathcal{Y}, \mathrm{Spc})$ and objects of $\mathrm{Fun}(\mathcal{Y}, \mathbf{Spt})$).

Remark 4.2.3. By Proposition 2.2.20, the constructions $\mathcal{T} \mapsto \mathrm{Mod}_{\mathcal{X}}(\mathcal{T})$ and $\mathcal{Y} \mapsto \mathrm{Str}_{\mathcal{A}}(\mathcal{Y})$ fit into a reflective adjunction

$$\mathrm{Str}_{\mathcal{A}} : \mathrm{Cat}_{/\mathcal{X}}^{\mathcal{A}\text{-thbl}} \rightleftarrows \mathrm{Thy}_{\mathcal{X}|\mathcal{A}}^{\mathrm{op}} : \mathrm{Mod}_{\mathcal{X}}$$

(where $\mathrm{Cat}_{/\mathcal{X}}^{\mathcal{A}\text{-thbl}} \subseteq \mathrm{Cat}_{/\mathcal{X}}$ is the full subcategory of $\mathcal{A}$ -theorizable ∞ -categories over $\mathcal{X}$). Reflectivity here means that $\mathcal{X}|\mathcal{A}$ -theories contravariantly embed into ∞ -categories over $\mathcal{X}$ via their ∞ -categories of models; we refer to objects in the essential image of this embedding as $\mathcal{A}$ -theoretical ∞ -categories over $\mathcal{X}$. For $\mathcal{Y} \in \mathrm{Cat}_{/\mathcal{X}}^{\mathcal{A}\text{-thbl}}$, we set

$$\mathcal{Y}^{\mathcal{A}\text{-th}/\mathcal{X}} := \mathrm{Mod}_{\mathcal{X}}(\mathrm{Str}_{\mathcal{A}}(\mathcal{Y}))$$

and refer to this as the $\mathcal{A}$ -theorization of $\mathcal{Y}$: the unit map $\eta : \mathcal{Y} \rightarrow \mathcal{Y}^{\mathcal{A}\text{-th}/\mathcal{X}}$ of the reflective adjunction above, which carries $Y \in \mathcal{Y}$ to the model of $\mathrm{Str}_{\mathcal{A}}(\mathcal{Y})$ given by evaluation at Y , is the initial map from $\mathcal{Y}$ to an $\mathcal{A}$ -theoretical ∞ -category over $\mathcal{X}$ (Definition 2.2.24 and Remark 2.2.25).

Example 4.2.4. Continuing Example 4.2.2, for $\mathcal{Y} \in \mathrm{Cat}_{/\mathbf{Spt}^I}$, a model of $\mathrm{Str}_{\mathbf{A}^I}(\mathcal{Y})$ in $\mathbf{Spt}^I$ is a functor $M : \mathrm{Str}_{\mathbf{A}^I}(\mathcal{Y}) \rightarrow \mathrm{Spc}$ that preserves finite products and the canonical pullback diagram

$$\begin{array}{ccc} \Omega^{\infty-n}u_i & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & \Omega^{\infty-n-1}u_i \end{array}$$

for each $i \in I$ and $n \in \mathbb{Z}_{\geq 0}$ (here we have invoked one of the characterizations of models from Remark 4.1.8).

We next restate the main results of §2.3, which are essentially about understanding the behavior of colimits in $\mathcal{A}$ -theoretical ∞ -categories over $\mathcal{X}$. For psychological aid, we'll do so using the following terminology.

Definition 4.2.5. Let $\mathcal{Y}$ be an $\mathcal{A}$ -theorizable ∞ -category over $\mathcal{X}$, let $\mathcal{T} := \mathrm{Str}_{\mathcal{A}}(\mathcal{Y}) \subseteq \mathrm{Fun}(\mathcal{Y}, \mathrm{Spc})$, and let $\chi : \mathcal{A}^{\mathrm{op}} \rightarrow \mathcal{T}$ be the structure map. Let $\mathcal{Z}$ be a cocomplete ∞ -category and let $v : \mathcal{Y} \rightarrow \mathcal{Z}$ be a functor. Assume that v is $\mathcal{X}|\mathcal{A}$ -derivable, in the sense of Definition 3.4.1: since $\mathcal{Z}$ admits colimits, this reduces

to the condition that the restricted Yoneda functor $j_v^\vee : \mathcal{Z} \rightarrow \text{Fun}(\mathcal{Y}, \text{Spc})$ admits a partial right adjoint $\Gamma_v : \mathcal{T}^{\text{op}} \rightarrow \mathcal{Z}$ (which holds for example if $\mathcal{Y}$ is $\mathcal{A}$ -small and $\mathcal{Z}$ admits limits; see Example 3.4.2). Then:

1. We say that v satisfies the $\mathcal{A}$ -theoretical Eilenberg–Moore formula if $\Gamma_v : \mathcal{T}^{\text{op}} \rightarrow \mathcal{Z}$ preserves the initial object and χ -suspensions.

In addition, assuming that $\mathcal{A}$ is closed under finite sums in $\mathcal{X}$:

2. We say that v satisfies the $\mathcal{A}$ -theoretical Künneth formula if $\Gamma_v : \mathcal{T}^{\text{op}} \rightarrow \mathcal{Z}$ preserves finite coproducts.
3. We say that v satisfies the $\mathcal{A}$ -theoretical Eilenberg–Moore bar formula if $\Gamma_v : \mathcal{T}^{\text{op}} \rightarrow \mathcal{Z}$ preserves χ -bar constructions.

Remark 4.2.6. The idea behind Definition 4.2.5 is that, in some cases, it is natural to think of objects of $\mathcal{T} \subseteq \text{Fun}(\mathcal{Y}, \text{Spc})$ as “spaces” of some kind, and to think of the functor $\Gamma_v : \mathcal{T}^{\text{op}} \rightarrow \mathcal{Z}$ as associating some kind of “cohomology algebras” to these “spaces”. What we have called the Eilenberg–Moore formula says that Γ_v carries certain loopings of “spaces” to suspensions of “algebras”. What we have called the Künneth formula says that Γ_v carries certain products of “spaces” to coproducts of “algebras”. Some care needs to be taken in interpreting this terminology in practice, as the meaning of coproduct (or pushout) in $\mathcal{Z}$ may or may not agree with the most natural meaning of (relative) tensor product in $\mathcal{Z}$; hopefully the mnemonic is nevertheless beneficial.

Remark 4.2.7. Suppose that we are in the situation of Definition 4.2.5 and that $\mathcal{A}$ is closed under finite sums in $\mathcal{X}$. Let $\mathcal{A}^\circ \subseteq \mathcal{A}$ be any full subcategory that contains 0, is closed under desuspensions, and generates $\mathcal{A}$ under finite sums. If v satisfies the $\mathcal{A}$ -theoretical Künneth formula, then the $\mathcal{A}$ -theoretical Eilenberg–Moore bar formula is equivalent to the $\mathcal{A}$ -theoretical Eilenberg–Moore formula, for which it suffices to check that $\Gamma_v : \mathcal{T}^{\text{op}} \rightarrow \mathcal{Z}$ preserves $\chi|_{(\mathcal{A}^\circ)^{\text{op}}}$ -suspensions.

Remark 4.2.8. In the situation of Definition 4.2.5, we recall also from Definition 3.4.1 that the $\mathcal{X}|\mathcal{A}$ -derived functor $v_\dagger : \mathcal{Y}^{\mathcal{A}\text{-th}/\mathcal{X}} \rightarrow \mathcal{Z}$ is defined to be the left Kan extension of $\Gamma_v : \mathcal{T}^{\text{op}} \rightarrow \mathcal{Z}$ along the Yoneda embedding $j_\mathcal{T}^\vee : \mathcal{T}^{\text{op}} \hookrightarrow \text{Mod}_\mathcal{X}(\mathcal{T}) = \mathcal{Y}^{\mathcal{A}\text{-th}/\mathcal{X}}$. By Proposition 4.1.11, v satisfies the $\mathcal{A}$ -theoretical Eilenberg–Moore formula if and only if $v_\dagger$ preserves colimits. And by Proposition 4.1.15, if $\mathcal{A}$ is closed under finite sums in $\mathcal{X}$, then v satisfies the $\mathcal{A}$ -theoretical Eilenberg–Moore bar formula if and only if $v_\dagger$ preserves sifted colimits.

Theorem 4.2.9. *Suppose given a commutative triangle of ∞ -categories*

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{v} & \mathcal{Z} \\ & \searrow u \quad \swarrow w & \\ & \mathcal{X} & \end{array}$$

in which u is $\mathcal{A}$ -theorizable and w is $\mathcal{A}$ -theoretical. Let

$$\begin{array}{ccccc} & & v & & \\ & & \curvearrowright & & \\ \mathcal{Y} & \xrightarrow{\eta} & \mathcal{Y}^{\mathcal{A}\text{-th}/\mathcal{X}} & \xrightarrow{v'} & \mathcal{Z} \\ & \searrow u \quad \downarrow u' \quad \swarrow w & & & \\ & & \mathcal{X} & & \end{array}$$

be the extended commutative diagram determined by the universal property of the $\mathcal{A}$ -theorization $\mathcal{Y}^{\mathcal{A}\text{-th}/\mathcal{X}}$. Recall from Proposition 3.4.3 that v is $\mathcal{X}|\mathcal{A}$ -derivable, so that Definition 4.2.5 applies. Then the following statements hold:

1. *If v is strongly algebraic, or if v is algebraic and $\mathcal{A}$ is closed under finite products in $\mathcal{X}$, then u is $\mathcal{A}$ -theoretical, i.e. η is an equivalence.*
2. *The functor v' preserves colimits if and only if v satisfies the $\mathcal{A}$ -theoretical Eilenberg–Moore formula.*
3. *The functor v' preserves sifted colimits if and only if v satisfies the $\mathcal{A}$ -theoretical Eilenberg–Moore bar formula.*

Proof. Let $h : \mathcal{X} \rightarrow \mathcal{Z}$ be the left adjoint to w and let $\mathcal{A}_\mathcal{Z} \subseteq \mathcal{Z}$ be the essential image of $\mathcal{A}$ under

h. By Proposition 3.2.12, u is $\mathcal{A}$ -theoretical if and only if v is $\mathcal{A}_{\mathcal{Z}}$ -theoretical. Thus, 1 follows from Theorem 2.3.10, while 2 and 3 follow from Theorem 2.3.11 in combination with Propositions 4.1.11 and 4.1.15 (or alternatively from Proposition 3.4.3 and Remark 4.2.8). $\square$

Remark 4.2.10. As noted in Remark 2.3.12, if the functor v' (resp. u') preserves sifted colimits, then $\eta : \mathcal{Y} \rightarrow \mathcal{Y}^{\mathcal{A}\text{-th}/\mathcal{X}}$ is an algebraization of $\mathcal{Y}$ over $\mathcal{Z}$ (resp. over $\mathcal{X}$), in the sense of Definition 2.3.3.

Remark 4.2.11. Suppose that we are in the situation of Theorem 4.2.9. Let $\chi : \mathcal{A}^{\text{op}} \rightarrow \text{Str}_{\mathcal{A}}(\mathcal{Y})$ be the structure map, and let $h : \mathcal{X} \rightarrow \mathcal{Z}$ be the left adjoint of w . We recall that Corollary 2.2.30 supplies a criterion for the functor v' to be an equivalence: it is so if and only if the objects in the essential image of $\mathcal{A}$ under h are v -corealized, in the sense of Definition 1.2.5, i.e. if and only if the unit map $h(A) \rightarrow \Gamma_v j_v^\vee(h(A)) \simeq \Gamma_v(\chi(A))$ in $\mathcal{Z}$ is an equivalence for each $A \in \mathcal{A}$.

§4.3. Stability. For $\mathcal{X}|\mathcal{A}$ as in Notation 4.0.1, we have the following basic class of $\mathcal{A}$ -theoretical ∞ -categories over $\mathcal{X}$.

Proposition 4.3.1. *For a functor $u : \mathcal{Y} \rightarrow \mathcal{X}$, the following conditions are equivalent:*

1. u is monadic and creates colimits;
2. u is algebraic and $\mathcal{Y}$ is stable;
3. u is $\mathcal{A}$ -theoretical and $\mathcal{Y}$ is stable

Proof. That $1 \Rightarrow 2$ is clear and that $2 \Rightarrow 3$ follows from Theorem 4.2.9.1. Suppose that 3 holds. Then u is monadic by Proposition 2.2.10 and preserves filtered colimits by Remark 4.1.9. Since $\mathcal{Y}$ is stable, the former implies that u is exact, so in fact u preserves all colimits, proving that 1 holds. $\square$

Example 4.3.2. As arose in our proof of Theorem 4.1.2, any compactly generated stable ∞ -category $\mathcal{Y}$ admits, for some set I , a functor $u : \mathcal{Y} \rightarrow \text{Spt}^I$ that satisfies the equivalent conditions of Proposition 4.3.1: we may take u to be the functor corepresented by a set of compact generators $\{Y_i\}_{i \in I}$ of $\mathcal{Y}$.

Let us articulate a special case of Theorem 4.2.9.2 that allows us to detect when the ∞ -category of models of an $\mathcal{X}|\mathcal{A}$ -theory is of the above type.

Definition 4.3.3. We say that an $\mathcal{X}|\mathcal{A}$ -theory $\chi : \mathcal{A}^{\text{op}} \rightarrow \mathcal{T}$ is *protostable* if $\mathcal{T}$ is a pointed ∞ -category and, for each $A \in \mathcal{A}$, the χ -looping diagram

$$\begin{array}{ccc} \chi(A) & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & \chi(A[-1]) \end{array}$$

in $\mathcal{T}$ is cocartesian (i.e. also a suspension diagram).

Proposition 4.3.4. *Let $\chi : \mathcal{A}^{\text{op}} \rightarrow \mathcal{T}$ be an $\mathcal{X}|\mathcal{A}$ -theory. Then $\text{Mod}_{\mathcal{X}}(\mathcal{T})$ is stable if and only if $\mathcal{T}$ is protostable.*

Proof. The only if direction follows from the definitions of stability and protostability and the existence of the embedding $j_{\mathcal{T}}^\vee : \mathcal{T}^{\text{op}} \hookrightarrow \text{Mod}_{\mathcal{X}}(\mathcal{T})$. For the if direction, it suffices by Proposition 4.3.1 to show that $\mathcal{T}$ being protostable implies that the structure map $u : \text{Mod}_{\mathcal{X}}(\mathcal{T}) \rightarrow \mathcal{X}$ preserves colimits. By Theorem 4.2.9.2, u preserves colimits if and only if the restriction of u along the Yoneda embedding $j_{\mathcal{T}}^\vee$ preserves the initial object and χ -suspensions. This holds by the protostability of $\mathcal{T}$: the case of the initial object follows from $\mathcal{T}$ and $\mathcal{X}$ being pointed and u preserving the final object; similarly, the case of χ -suspensions follows from the χ -looping diagrams in $\mathcal{T}$ being cocartesian, $j_{\mathcal{T}}^\vee$ and u both preserving pullbacks, and the stability of $\mathcal{X}$. $\square$

Remark 4.3.5. It follows from Proposition 4.3.4 that a protostable $\mathcal{X}|\mathcal{A}$ -theory $\mathcal{T}$ is not only pointed but additive.

Protostable theories will arise in Chs. 5 and 6 by the following mechanism.

Proposition 4.3.6. *Let $u : \mathcal{Y} \rightarrow \mathcal{X}$ be an $\mathcal{A}$ -theorizable functor. Suppose that the following conditions are satisfied:*

1. *the ∞ -category $\mathcal{Y}$ and the functor u are additive;*
2. *$\text{Ext}_{\mathcal{X}}^k(A, u(Y)) \simeq 0$ for all $A \in \mathcal{A}$, $Y \in \mathcal{Y}$, and $k > 0$.*

Then the associated $\mathcal{X}|\mathcal{A}$ -theory $\chi : \mathcal{A}^{\text{op}} \rightarrow \text{Str}_{\mathcal{A}}(\mathcal{Y})$ is protostable.

Proof. Set $\mathcal{T} := \text{Str}_{\mathcal{A}}(\mathcal{Y})$. It follows from 1 that $\mathcal{T}$ lies in the full subcategory $\text{Fun}'(\mathcal{Y}, \text{Spc}) \subseteq \text{Fun}(\mathcal{Y}, \text{Spc})$ spanned by those functors that preserve finite products. By [29, Proposition C.1.5.7], the additivity of $\mathcal{Y}$ implies that $\text{Fun}'(\mathcal{Y}, \text{Spc})$ is a prestable ∞ -category, embedding inside its stabilization, which identifies with the ∞ -category $\text{Fun}'(\mathcal{Y}, \text{Spt})$ of finite product-preserving functors from $\mathcal{Y}$ to Spt . This immediately implies that $\mathcal{T}$ is pointed. Regarding the χ -looping diagram for $A \in \mathcal{A}$, when viewed in $\text{Fun}'(\mathcal{Y}, \text{Spt})$, it may be identified as follows:

$$\begin{array}{ccc} \tau_{\geq 0} \underline{\text{Map}}_{\mathcal{X}}(A, u(Y)) & \xrightarrow{\quad} & 0 \\ \downarrow & & \downarrow \\ 0 & \xrightarrow{\quad} & \tau_{\geq 0}(\underline{\text{Map}}_{\mathcal{X}}(A, u(Y))[1]). \end{array}$$

It follows from 2 that the connective covers are redundant, from which we see that the above square is a pushout diagram in $\text{Fun}'(\mathcal{Y}, \text{Spt})$, hence also in the full subcategory $\mathcal{T}$. $\square$

§4.4. Connectivity. With $\mathcal{X}$ and $\mathcal{A}$ still as in Notation 4.0.1, our aim in this section is to examine how $\mathcal{X}|\mathcal{A}$ -theories and their models interact with connectivity with respect to a t-structure on $\mathcal{X}$. We will focus on a particularly simple situation, corresponding to a special class of t-structures; let's begin by reviewing this setting.

Lemma 4.4.1. *Let $\mathcal{A}_0 \subseteq \mathcal{X}$ be a full subcategory and let $\mathcal{X}_{\geq 0} \subseteq \mathcal{X}$ be the full subcategory generated under colimits and extensions by $\mathcal{A}_0$. Let A be a compact object of $\mathcal{X}$. Then the following conditions are equivalent:*

1. *$\text{Ext}_{\mathcal{X}}^k(A, A') \simeq 0$ for any $A' \in \mathcal{A}_0$ and $k > 0$;*
2. *$\text{Ext}_{\mathcal{X}}^k(A, X) \simeq 0$ for any $X \in \mathcal{X}_{\geq 0}$ and $k > 0$;*
3. *the functor $\text{Map}_{\mathcal{X}}(A, -) : \mathcal{X}_{\geq 0} \rightarrow \text{Spc}$ preserves sifted colimits.*

Proof. The condition that $\text{Ext}_{\mathcal{X}}^k(A, X) \simeq 0$ for $k > 0$ is equivalent to the mapping spectrum $\underline{\text{Map}}_{\mathcal{X}}(A, X)$ being connective. The collection of $X \in \mathcal{X}$ for which this is satisfied is closed under colimits and extensions, because A is compact in $\mathcal{X}$, which implies that $\underline{\text{Map}}_{\mathcal{X}}(A, -) : \mathcal{X} \rightarrow \text{Spt}$ preserves colimits, and connectivity is closed under colimits and extensions. This shows that $1 \Rightarrow 2$, and the reverse implication is tautological.

We now show the equivalence between 2 and 3. Assume first that 2 holds. Then the functor $\underline{\text{Map}}_{\mathcal{X}}(A, -) : \mathcal{X} \rightarrow \text{Spt}$, which preserves colimits by the compactness hypothesis, restricts to a functor $\underline{\text{Map}}_{\mathcal{X}}(A, -) : \mathcal{X}_{\geq 0} \rightarrow \text{Spt}_{\geq 0}$, which consequently also preserves colimits. Since $\Omega^{\infty} : \text{Spt}_{\geq 0} \rightarrow \text{Spc}$ preserves sifted colimits [26, Corollary 5.2.6.7] and $\text{Map}_{\mathcal{X}}(A, -) \simeq \Omega^{\infty} \underline{\text{Map}}_{\mathcal{X}}(A, -)$, we deduce 3.

Assume now that 3 holds. Reversing the final step from the previous paragraph, we find that $\tau_{\geq 0} \underline{\text{Map}}_{\mathcal{X}}(A, -) : \mathcal{X}_{\geq 0} \rightarrow \text{Spt}_{\geq 0}$ preserves sifted colimits; since it's also an additive functor, it therefore preserves all colimits. In particular, for any $X \in \mathcal{X}_{\geq 0}$ and $k > 0$, we have

$$\tau_{\geq 0} \underline{\text{Map}}_{\mathcal{X}}(A, X[k]) \simeq (\tau_{\geq 0} \underline{\text{Map}}_{\mathcal{X}}(A, X))[k];$$

applying π_0 to this gives $\text{Ext}_{\mathcal{X}}^k(A, X) \simeq 0$, proving 2. $\square$

Notation 4.4.2. For the remainder of the section, we suppose given a full subcategory $\mathcal{A}_0 \subseteq \mathcal{A}$ that generates $\mathcal{A}$ under desuspensions that satisfies the following additional hypotheses:

1. $\text{Ext}_{\mathcal{X}}^k(A, A') \simeq 0$ for all $A, A' \in \mathcal{A}_0$ and $k > 0$;
2. $\mathcal{A}_0$ is closed under finite sums in $\mathcal{X}$ (implying that $\mathcal{A}$ is too).

We let $\mathcal{X}_{\geq 0} \subseteq \mathcal{X}$ be the full subcategory generated under colimits and extensions by $\mathcal{A}_0$ and we let $\mathcal{X}_{\leq 0} \subseteq \mathcal{X}$ be the full subcategory spanned by those objects X such that $\text{Ext}_{\mathcal{X}}^k(A, X) \simeq 0$ for $A \in \mathcal{A}_0$ and $k < 0$. By [29, Proposition C.6.3.1] and our hypothesis that $\mathcal{A}$ is a detecting subcategory of compact objects, $(\mathcal{X}_{\geq 0}, \mathcal{X}_{\leq 0})$ is a right complete t-structure on $\mathcal{X}$. We write $\iota : \mathcal{X}_{\geq 0} \rightarrow \mathcal{X}$ for the inclusion functor.

Example 4.4.3. In the case $\mathcal{X} = \text{Spt}$ and $\mathcal{A} = \mathbf{A}$ (Example 4.0.2), we may take $\mathcal{A}_0$ to be the full subcategory $\mathcal{A}_0$ generated under finite sums by $\mathbf{S}$. The resulting t-structure on Spt is the standard one.

More generally, if $\mathcal{X} = \text{Spt}^I$ and $\mathcal{A} = \mathbf{A}^I$ for a set I , then we may take $\mathcal{A}_0 = \mathbf{A}_0^I$, i.e. the full subcategory generated under finite sums by $\mathbf{S}(-i)$ for $i \in I$, and the resulting t-structure on Spt^I is given pointwise by the one of the previous paragraph.

Proposition 4.4.4. *The following statements hold:*

1. *The restricted Yoneda functor $j_{\mathcal{X}_{\geq 0}|\mathcal{A}_0} : \mathcal{X}_{\geq 0} \rightarrow \mathcal{P}_{\Sigma}(\mathcal{A}_0)$ is an equivalence.*
2. *For any $A \in \mathcal{A}$, the functor $\text{Map}_{\mathcal{X}}(A, -) : \mathcal{X}_{\geq 0} \rightarrow \text{Spc}$ preserves sifted colimits.*
3. *For each $X \in \mathcal{X}_{\geq 0}$, the presheaf $j_{\mathcal{X}|\mathcal{A}}(X) \in \mathcal{P}(\mathcal{A})$ is left Kan extended from $\mathcal{A}_0$. In other words, the embedding*

$$\iota : \mathcal{X}_{\geq 0}|\mathcal{A}_0 \hookrightarrow \mathcal{X}|\mathcal{A}$$

in Prd^L is strict, in the sense of Definition 3.3.7.

Proof. Since $\mathcal{A}$ is a detecting subcategory of $\mathcal{X}$ and $\mathcal{A}_0$ generates $\mathcal{A}$ under desuspensions, the functors $\text{Map}_{\mathcal{X}}(A, -) : \mathcal{X} \rightarrow \text{Spt}$ for $A \in \mathcal{A}_0$ are jointly conservative. By the projectivity hypothesis, these functors carry $\mathcal{X}_{\geq 0}$ to $\text{Spt}_{\geq 0}$. Since $\Omega^{\infty} : \text{Spt}_{\geq 0} \rightarrow \text{Spc}$ is conservative, it follows that $\mathcal{A}_0$ is a detecting subcategory of $\mathcal{X}_{\geq 0}$. The objects of $\mathcal{A}_0$ are also compact projective in $\mathcal{X}_{\geq 0}$ by projectivity, so 1 now follows from Proposition 2.3.17.

Assertion 2 is a consequence of projectivity and the fact that the equivalent conditions of Lemma 4.4.1 are closed under desuspensions in $\mathcal{A}$. Note that 2 may be reformulated as the assertion that the restriction of $j_{\mathcal{X}|\mathcal{A}} : \mathcal{X} \rightarrow \mathcal{P}(\mathcal{A})$ to $\mathcal{X}_{\geq 0}$ preserves sifted colimits. From this and the fact that $\mathcal{X}_{\geq 0}$ is generated under sifted colimits by $\mathcal{A}_0$ (the latter a consequence of 1), we deduce 3, since the condition of being left Kan extended from a full subcategory is stable under colimits. $\square$

Remark 4.4.5. By Proposition 4.4.4.1, it makes sense to consider $\mathcal{X}_{\geq 0}|\mathcal{A}_0$ -theories $\chi : \mathcal{A}_0^{\text{op}} \rightarrow \mathcal{T}$. In fact, an $\mathcal{X}_{\geq 0}|\mathcal{A}_0$ -theory is simply an essentially surjective, finite product preserving functor $\chi : \mathcal{A}_0^{\text{op}} \rightarrow \mathcal{T}$ where $\mathcal{T}$ is locally small; and a model thereof in $\mathcal{X}_{\geq 0}$ is simply a finite product preserving functor $M : \mathcal{T} \rightarrow \text{Spc}$ (Remark 2.3.13).

In the rest of the section, we review some material from §3.3 in the context of the strict embedding $\iota : \mathcal{X}_{\geq 0}|\mathcal{A}_0 \rightarrow \mathcal{X}|\mathcal{A}$ (Proposition 4.4.4.3).

Notation 4.4.6. Let $\chi : \mathcal{A}^{\text{op}} \rightarrow \mathcal{T}$ be an $\mathcal{X}|\mathcal{A}$ -theory, and let $u : \text{Mod}_{\mathcal{X}}(\mathcal{T}) \rightarrow \mathcal{X}$ be the structure map.

1. We will sometimes write $\chi_0 : \mathcal{A}_0^{\text{op}} \rightarrow \mathcal{T}_0$ for the restriction along ι of $\mathcal{T}$, in the sense of Construction 3.1.7. That is, $\mathcal{T}_0$ denotes the essential image of $\mathcal{A}_0^{\text{op}}$ under χ , and χ_0 is obtained by restricting χ . This is an $\mathcal{X}_{\geq 0}|\mathcal{A}_0$ -theory is clear in the present situation and also a special case of Proposition 3.1.8.2.
2. We denote by $\text{Mod}_{\mathcal{X}}(\mathcal{T})_{\geq 0} \subseteq \text{Mod}_{\mathcal{X}}(\mathcal{T})$ the full subcategory spanned by those models $M : \mathcal{T} \rightarrow \text{Spc}$ such that $u(M) \in \mathcal{X}_{\geq 0}$.

Remark 4.4.7. Let $\chi : \mathcal{A}^{\text{op}} \rightarrow \mathcal{T}$ be an $\mathcal{X}|\mathcal{A}$ -theory. Beware that it may or may not be true that the Yoneda embedding $j_{\mathcal{T}}^{\vee} : \mathcal{T} \hookrightarrow \text{Mod}_{\mathcal{X}}(\mathcal{T})$ carries the restriction $\mathcal{T}_0$ into $\text{Mod}_{\mathcal{X}}(\mathcal{T})_{\geq 0}$. We have a name for this condition: it is true if and only if $\mathcal{T}$ is ι -consistent (Definition 3.3.10).

We recall also the complementary condition of $\mathcal{T}$ being ι -extended, meaning that the structure map χ is left Kan extended from $\mathcal{A}_0^{\text{op}}$ (Definition 3.3.15), or equivalently that the restriction of the associated monad $t : \mathcal{X} \rightarrow \mathcal{X}$ to $\mathcal{A}$ is right Kan extended from $\mathcal{A}_0$ (Lemma 3.3.18).

Example 4.4.8. Let $\mathcal{Y} \in \text{Cat}/\mathcal{X}_{\geq 0}$ and assume that $\mathcal{Y}$ is $\mathcal{A}$ -theorizable as an ∞ -category over $\mathcal{X}$. Then the associated $\mathcal{X}|\mathcal{A}$ -theory $\text{Str}_{\mathcal{A}}(\mathcal{Y}) \subseteq \text{Fun}(\mathcal{Y}, \text{Spc})$ is ι -extended, by Lemma 3.3.17.1.

Proposition 4.4.9. Let $\chi : \mathcal{A}^{\text{op}} \rightarrow \mathcal{T}$ be an ι -consistent $\mathcal{X}|\mathcal{A}$ -theory, and let $\chi_0 : \mathcal{A}_0^{\text{op}} \rightarrow \mathcal{T}_0$ be its restriction along ι . Then:

1. left Kan extension along the inclusion functor $\mathcal{T}_0 \hookrightarrow \mathcal{T}$ defines an equivalence of ∞ -categories $\theta : \text{Mod}_{\chi_{\geq 0}}(\mathcal{T}_0) \simeq \text{Mod}_{\chi}(\mathcal{T})_{\geq 0}$ over $\mathcal{X}_{\geq 0}$, so that we have a cartesian square

$$\begin{array}{ccc} \text{Mod}_{\chi_{\geq 0}}(\mathcal{T}_0) & \xhookrightarrow{\theta} & \text{Mod}_{\chi}(\mathcal{T}) \\ u_{\geq 0} \downarrow & & \downarrow u \\ \mathcal{X}_{\geq 0} & \xrightarrow{\iota} & \mathcal{X} \end{array}$$

(the vertical arrows being the structure maps), which furthermore is horizontally right adjointable (equivalently vertically left adjointable);

2. the inclusion $\text{Mod}_{\chi}(\mathcal{T})_{\geq 0} \hookrightarrow \text{Mod}_{\chi}(\mathcal{T})$ identifies the target as the $\mathcal{A}$ -theorization of the source over $\mathcal{X}$ if and only if $\mathcal{T}$ is ι -extended.

Proof. The first claim is a special case of Proposition 3.3.11.2, and the second then follows from Proposition 3.3.16.1. $\square$

Remark 4.4.10. In a similar spirit to Proposition 4.4.9.2, the recognition criterion for $\mathcal{A}$ -theorizations stated in Remark 4.2.11 may be simplified under connectivity hypotheses. Namely, in the situation of Theorem 4.2.9, assuming that the essential image of u is contained in $\mathcal{X}_{\geq 0}$ and that the $\mathcal{X}|\mathcal{A}$ -theory $\text{Str}_{\mathcal{A}}(\mathcal{Z})$ is ι -extended, the functor v' is an equivalence if and only if the canonical map $h(A) \rightarrow \Gamma_v(X(A))$ in $\mathcal{Z}$ is an equivalence for each $A \in \mathcal{A}_0$; this is a special case of Remark 3.3.19.

It will be worth understanding the notion of ι -extendedness more explicitly. It boils down to the following dual version of Definition 4.1.13.

Definition 4.4.11. Let $\mathcal{E}$ be any ∞ -category. We say that a functor $f : \mathcal{A} \rightarrow \mathcal{E}$ *preserves cobar constructions* if, for each $A, B \in \mathcal{A}$, it preserves the totalization

$$\lim_{\Delta} (B \oplus \text{cobar}_A(0, 0)^{\bullet}) \simeq B \oplus A[-1].$$

Dually, we say that a functor $f : \mathcal{A}^{\text{op}} \rightarrow \mathcal{E}$ *preserves bar constructions* if $f^{\text{op}} : \mathcal{A} \rightarrow \mathcal{E}^{\text{op}}$ preserves cobar constructions.

Proposition 4.4.12. Let $\mathcal{E}$ be any ∞ -category and let $f : \mathcal{A} \rightarrow \mathcal{E}$ be a functor. Then f is right Kan extended from $\mathcal{A}_0$ if and only if it preserves cobar constructions.

Proof. This reduces to the claim that the restricted Yoneda functor $j_{\mathcal{A}|\mathcal{A}_0}^{\vee} : \mathcal{A} \rightarrow \text{Fun}(\mathcal{A}_0, \text{Spc})^{\text{op}}$ preserves cobar constructions. Consider the spectral variant

$$\underline{j}_{\mathcal{A}|\mathcal{A}_0}^{\vee} : \mathcal{A} \rightarrow \text{Fun}(\mathcal{A}_0, \text{Spt})^{\text{op}}, \quad A \mapsto \underline{\text{Map}}_{\mathcal{X}}(A, -);$$

this functor is additive and preserves loopings, hence preserves cobar constructions. Since $\underline{\text{Map}}_{\mathcal{X}}(A, A')$ is connective for any $A \in \mathcal{A}$ and $A' \in \mathcal{A}_0$ (recalling our hypotheses from Notation 4.4.2) and $\Omega^{\infty} : \text{Spt}_{\geq 0} \rightarrow \text{Spc}$ preserves geometric realizations, this implies the claim. $\square$

Corollary 4.4.13. Let $\chi : \mathcal{A}^{\text{op}} \rightarrow \mathcal{T}$ be an $\mathcal{X}|\mathcal{A}$ -theory and let $t : \mathcal{X} \rightarrow \mathcal{X}$ be the monad associated to the structure map $u : \text{Mod}_{\chi}(\mathcal{T}) \rightarrow \mathcal{X}$. Then the following conditions are equivalent:

1. $\mathcal{T}$ is ι -extended;
2. χ preserves bar constructions;
3. $t|_{\mathcal{A}}$ preserves cobar constructions.

Remark 4.4.14. Let us take note of one more way to think about condition 2 of Corollary 4.4.13. For this we first review some generalities.

For an ∞ -category $\mathcal{E}$ that admits finite products, let us write $\mathbf{CMon}^{\mathrm{gp}}(\mathcal{E})$ for the ∞ -category of grouplike commutative monoid objects of $\mathcal{E}$. The ∞ -category $\mathbf{CMon}^{\mathrm{gp}}(\mathcal{E})$ is additive. The forgetful functor $\mathbf{CMon}^{\mathrm{gp}}(\mathcal{E}) \rightarrow \mathcal{E}$ creates limits, in particular preserves finite products, and it is an equivalence if (and only if) $\mathcal{E}$ is additive. Given maps $B \leftarrow A \rightarrow C$ in $\mathbf{CMon}^{\mathrm{gp}}(\mathcal{E})$, let us write $\mathrm{bar}_A^\times(B, C)_\bullet$ for the associated bar construction in $\mathbf{CMon}^{\mathrm{gp}}(\mathcal{E})$ (see Remark 4.1.12; the additional decoration $\times$ here is meant to distinguish from a bar construction in $\mathcal{E}$, indicating that the coproduct in $\mathbf{CMon}^{\mathrm{gp}}(\mathcal{E})$ is given by the product in $\mathcal{E}$). Below, we will be interested in the case that B and C are both a final object $1 \in \mathcal{E}$. We write a colimit of $\mathrm{bar}_A^\times(1, 1)_\bullet$ by BA ; in the case $\mathcal{E} = \mathbf{Spc}$, this is referred to as the *classifying space* construction. Such a colimit is equivalent to a suspension of A in $\mathbf{CMon}^{\mathrm{gp}}(\mathcal{E})$, so any commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & A' \end{array}$$

in $\mathbf{CMon}^{\mathrm{gp}}(\mathcal{E})$ (in particular an equivalence $A \simeq \Omega A'$) induces a map $BA \rightarrow A'$, assuming that the colimit exists, and more generally induces an augmentation $\mathrm{bar}_A^\times(1, 1)_\bullet \rightarrow A'$.

Now let $\chi : \mathcal{A}^{\mathrm{op}} \rightarrow \mathcal{T}$ be an $\mathcal{X}|\mathcal{A}$ -theory. Note that $\mathcal{A}$ is additive and that χ preserves finite products. It follows that the latter canonically factors through the forgetful functor $\mathbf{CMon}^{\mathrm{gp}}(\mathcal{T}) \rightarrow \mathcal{T}$. In particular, for each $A \in \mathcal{A}$, the object $\chi(A)$ in $\mathcal{T}$ carries a canonical grouplike commutative monoid structure; furthermore, the equivalence $\chi(A) \simeq \Omega(\chi(A[-1]))$ respects this structure, and hence determines an augmented simplicial object

$$T \times \mathrm{bar}_{\chi(A)}^\times(1, 1)_\bullet \rightarrow T \times \chi(A[-1])$$

for any $T \in \mathcal{T}$. The condition that χ preserve bar constructions, which is equivalent to $\mathcal{T}$ being ι -extended by Corollary 4.4.13, is the condition that the above augmented simplicial diagrams be colimit diagrams. In particular, taking T to be a final object, this condition implies that we have equivalences $B\chi(A) \simeq \chi(A[-1])$ for $A \in \mathcal{A}$.

Example 4.4.15. Example 4.4.8 can also be understood using the explicit criterion for ι -extendedness in Remark 4.4.14. Let $u : \mathcal{Y} \rightarrow \mathcal{X}_{\geq 0}$ and $\chi : \mathcal{A}^{\mathrm{op}} \rightarrow \mathbf{Str}_{\mathcal{A}}(\mathcal{Y})$ be the structure maps. For $Y \in \mathcal{Y}$ and $A \in \mathcal{A}$, the canonical map of spaces

$$B(\chi(A)(Y)) \simeq B(\mathrm{Map}_{\mathcal{X}}(A, u(Y))) \rightarrow \mathrm{Map}_{\mathcal{X}}(A, u(Y)[1]) \simeq \chi(A[-1])(Y)$$

is an equivalence, because $u(Y)$ lying in $\mathcal{X}_{\geq 0}$ implies that the spectrum $\underline{\mathrm{Map}}_{\mathcal{X}}(A, u(Y))$ is connective (Lemma 4.4.1.2). Together with the fact that finite products distribute over colimits in $\mathbf{Spc}$, this verifies the criterion.

We conclude with an addendum to Proposition 4.4.9 in the setting of the previous section.

Proposition 4.4.16. *Let $\mathcal{T}$ be a protostable $\mathcal{X}|\mathcal{A}$ -theory. Then:*

1. *if $\mathcal{T}$ is ι -consistent, $\mathrm{Mod}_{\mathcal{X}}(\mathcal{T})_{\geq 0}$ is the connective part of a right complete t -structure on $\mathrm{Mod}_{\mathcal{X}}(\mathcal{T})$;*
2. *$\mathcal{T}$ is ι -extended.*

Proof. Claim 1 is straightforward from Proposition 4.4.9.1, which, under the current hypotheses, provides an identification $\mathrm{Mod}_{\mathcal{X}}(\mathcal{T})_{\geq 0} \simeq \mathcal{P}_{\Sigma}^{\vee}(\mathcal{T}_0)$. Claim 2 follows from the characterization of ι -extendedness in Corollary 4.4.13, along with the definition of protostability and Remark 4.3.5. $\square$

Example 4.4.17. By Proposition 4.4.16.2 and Proposition 4.4.9.2, the inclusion functor $\mathcal{X}_{\geq 0} \hookrightarrow \mathcal{X}$ exhibits $\mathcal{X}$ as the $\mathcal{A}$ -theorization of $\mathcal{X}_{\geq 0}$ over $\mathcal{X}$.

Ch. 5. Bianimation

Let $\mathbf{Ab}$ denote the category of abelian groups and let $D(\mathbf{Ab})$ denote the derived ∞ -category of complexes of abelian groups. The full subcategory $D(\mathbf{Ab})_{\geq 0} \subseteq D(\mathbf{Ab})$ spanned by the connective complexes has an alternative description: it is the ∞ -category $\mathbf{Ani}(\mathbf{Ab})$ of animated abelian groups.

The animation construction $\mathbf{Ani}(-)$, which we will review in §5.1, is a reformulation of Quillen’s theory of nonabelian derived categories. As the latter terminology suggests, the construction has the feature of applying not just to abelian categories. For example, any algebraic functor (Definition 2.3.1) $u : \mathcal{Y} \rightarrow \mathbf{Ab}$ may be animated, giving an algebraic functor $\mathbf{Ani}(u) : \mathbf{Ani}(\mathcal{Y}) \rightarrow \mathbf{Ani}(\mathbf{Ab}) \simeq D(\mathbf{Ab})_{\geq 0}$ which fits into a cartesian square of ∞ -categories

$$\begin{array}{ccc} \mathcal{Y} & \hookrightarrow & \mathbf{Ani}(\mathcal{Y}) \\ u \downarrow & & \downarrow \mathbf{Ani}(u) \\ \mathbf{Ab} & \hookrightarrow & D(\mathbf{Ab})_{\geq 0} \end{array}$$

(the lower horizontal arrow being the embedding of the heart). One case of particular interest is where $\mathcal{Y}$ is the category $\mathbf{CRing}$ of commutative rings (and u is the forgetful functor); just as commutative rings are the affine building blocks of algebraic geometry, animated commutative rings form the affine building blocks of derived algebraic geometry.

Do these ideas extend beyond the connective setting? To be more precise, we could ask:

Question 5.0.1. *Given an algebraic functor $u : \mathcal{Y} \rightarrow \mathbf{Ab}$, does there exist an algebraic functor $u : \mathfrak{Y} \rightarrow D(\mathbf{Ab})$ such that the previous cartesian square extends by another one,*

$$\begin{array}{ccc} \mathbf{Ani}(\mathcal{Y}) & \hookrightarrow & \mathfrak{Y} \\ \mathbf{Ani}(u) \downarrow & & \downarrow u \\ D(\mathbf{Ab})_{\geq 0} & \hookrightarrow & D(\mathbf{Ab}) \end{array}$$

(where the lower horizontal arrow is the inclusion functor)?

If $\mathcal{Y}$ is abelian, then we again have $\mathbf{Ani}(\mathcal{Y}) \simeq D(\mathcal{Y})_{\geq 0}$, and we may answer the question positively by taking $\mathfrak{Y} = D(\mathcal{Y})$, which can also be characterized as the stabilization of $\mathbf{Ani}(\mathcal{Y})$. But the situation is less clear if $\mathcal{Y}$ is not abelian; in particular, we cannot simply stabilize, since $\mathbf{Ani}(\mathcal{Y})$ does not embed in its stabilization.

A more general positive answer to Question 5.0.1—still requiring a hypothesis on u , but one which accommodates a large class of non-abelian $\mathcal{Y}$ —comes from work of Brantner–Mathew [12]. In the case $\mathcal{Y} = \mathbf{CRing}$, where Question 5.0.1 naturally arises in understanding the structure of the derived global sections of a non-affine derived scheme or stack, an account of this answer is given in [35, §4], where the ∞ -category $\mathfrak{Y}$ is denoted by $D\mathbf{Alg}$ and its objects are called derived commutative rings. More general accounts may be found in work of Brantner–Campos–Nuiten [10, §2] and Brantner–Magidson–Nuiten [11, §A]. The approach is monadic: Brantner–Mathew study a canonical mechanism for extending the monads associated to u and $\mathbf{Ani}(u)$ to a monad on the full derived ∞ -category $D(\mathbf{Ab})$.

In this chapter, we’ll discuss an alternative perspective on the Brantner–Mathew construction based on the ideas developed earlier in this paper, i.e. emphasizing theories instead of monads. In this way, we’ll get an explicit presentation of the resulting ∞ -category $\mathfrak{Y}$ and functor u , in addition to their associated monad. Furthermore, our results on algebraization will imply that this solution to Question 5.0.1 is a universal one.

First, in §5.1, we’ll review the theory of animation and explain its connection to Ch. 2. This connection, together with our work in Ch. 4, suggests a nonconnective analogue; we’ll spell this out in §5.2, formulating a construction which we term *bianimation*, and we’ll then explain how this recovers the construction of Brantner–Mathew answering Question 5.0.1. We’ll then review a couple of special cases of bianimation: in §5.3, we’ll show that it recovers the derived ∞ -category $D(\mathcal{Y})$ for abelian $\mathcal{Y}$ as above; and in §5.4, we’ll

revisit the ∞ -category of DAlg of derived commutative rings from this perspective (proving Theorems 0.1.1 and 0.1.4 from the introduction). Finally, in §5.5, we'll discuss how to recover $\mathrm{D}(\mathcal{Y})$ for a different class of abelian categories $\mathcal{Y}$ using a variant of biamination.

§5.1. Animation. Let us first briefly review the construction as it is explained in [14, §5.1]. The context is the following:

Definition 5.1.1. Let $\mathcal{X}$ be a category. We say that $\mathcal{X}$ is *classically algebraic* if it is cocomplete and locally small and the full subcategory $\mathcal{A} \subseteq \mathcal{X}$ spanned by the compact 1-projective objects is essentially small and detecting (Convention 0.3.11).

Example 5.1.2. For any set I , the category Set^I is classically algebraic, its full subcategory of compact 1-projective objects being Σ^I (cf. Remark 2.3.13).

Remark 5.1.3. Let $\mathcal{X}$ be a cocomplete, locally small category. Then $\mathcal{X}$ is classically algebraic if and only if it admits an essentially small detecting subcategory of compact 1-projective objects $\mathcal{A}' \subseteq \mathcal{X}$. The “only if” direction is tautological, and the “if” direction is a consequence of Remark 2.3.19, which says that the full subcategory $\mathcal{A} \subseteq \mathcal{X}$ of all compact 1-projective objects is generated under finite coproducts and retracts by $\mathcal{A}'$.

Remark 5.1.4. Let $\mathcal{X}$ and $\mathcal{Y}$ be cocomplete, locally small categories and let $u : \mathcal{Y} \rightarrow \mathcal{X}$ be a functor. If u preserves sifted colimits and admits a left adjoint $f : \mathcal{X} \rightarrow \mathcal{Y}$, then f preserves compact 1-projective objects.

Now suppose that $\mathcal{X}$ is algebraic, and let $\mathcal{A} \subseteq \mathcal{X}$ be the full subcategory of compact 1-projective objects. Then, if u is algebraic, the essential image of $\mathcal{A}$ under the left adjoint $f : \mathcal{X} \rightarrow \mathcal{Y}$ of u compact 1-projectively generates $\mathcal{Y}$, which by Remark 5.1.3 implies that $\mathcal{Y}$ is classically algebraic.

Remark 5.1.5. All classically algebraic categories arise via the combination of Example 5.1.2 and Remark 5.1.4: for $\mathcal{X}$ a cocomplete, locally small category and I a set, an algebraic functor $u : \mathcal{X} \rightarrow \mathrm{Set}^I$ is equivalent to a choice of detecting set of compact 1-projective objects $\{X_i\}_{i \in I}$ in $\mathcal{X}$.

Animation is the following simple construction in this context.

Definition 5.1.6. Let $\mathcal{X}$ be a classically algebraic category. Letting $\mathcal{A} \subseteq \mathcal{X}$ be the full subcategory spanned by the compact 1-projective objects, we set

$$\mathrm{Ani}(\mathcal{X}) := \mathcal{P}_\Sigma(\mathcal{A})$$

and refer to this ∞ -category as the *animation* of $\mathcal{X}$. We use the alternative notation $\eta_{\mathrm{Ani}} : \mathcal{X} \rightarrow \mathrm{Ani}(\mathcal{X})$ for the restricted Yoneda functor. By Proposition 2.3.17, η_{Ani} is fully faithful and its essential image consists of the 0-truncated objects of $\mathrm{Ani}(\mathcal{X})$. We may refer to η_{Ani} as the *canonical embedding* of $\mathcal{X}$ into its animation.

Remark 5.1.7. Let $\mathcal{X}$ be a cocomplete, locally small category and let $\mathcal{A}' \subseteq \mathcal{X}$ be an essentially small, detecting subcategory of compact 1-projective objects that is closed under finite coproducts. By Remark 5.1.3, $\mathcal{X}$ is classically algebraic, and, letting $\mathcal{A} \subseteq \mathcal{X}$ again denote the full subcategory of all compact 1-projective objects, the restriction functor $\mathrm{Ani}(\mathcal{X}) = \mathcal{P}_\Sigma(\mathcal{A}) \rightarrow \mathcal{P}_\Sigma(\mathcal{A}')$ is an equivalence.

Remark 5.1.8. For $\mathcal{X}$ a classically algebraic category, the functor $\eta_{\mathrm{Ani}} : \mathcal{X} \rightarrow \mathrm{Ani}(\mathcal{X})$ admits a left adjoint; we denote it by $\pi_0 : \mathrm{Ani}(\mathcal{X}) \rightarrow \mathcal{X}$. If we identify $\mathcal{X}$ with its essential image under η_{Ani} , then this left adjoint is given by composition with $\pi_0 : \mathrm{Spc} \rightarrow \mathrm{Set}$ (which preserves finite products).

We note the following illustrative examples of animation, the first positioning it as a kind of “connective derived ∞ -category” construction, and the second being the most basic example, in view of Remark 5.1.5.

Example 5.1.9. Let $\mathcal{X}$ be an abelian, classically algebraic category. Let $\mathrm{D}(\mathcal{X})$ be its derived ∞ -category [26, Definition 1.3.5.8], which carries a canonical t-structure $(\mathrm{D}(\mathcal{X})_{\geq 0}, \mathrm{D}(\mathcal{X})_{\leq 0})$ whose heart identifies with $\mathcal{X}$ [26, Remark 1.3.5.19 and Proposition 1.3.5.21]. The resulting embedding $\mathcal{X} \hookrightarrow \mathrm{D}(\mathcal{X})_{\geq 0}$ extends uniquely to

an equivalence $\text{Ani}(\mathcal{X}) \simeq \text{D}(\mathcal{X})_{\geq 0}$ [26, Propositions 1.3.3.14 and 1.3.5.24].

Example 5.1.10. Let $\mathbf{I}$ be a set. Then we have a canonical equivalence $\text{Ani}(\text{Set}^{\mathbf{I}}) \simeq \text{Spc}^{\mathbf{I}}$.

Let us now recall one type of functoriality exhibited by animation.

Construction 5.1.11. Let $\mathcal{X}$ and $\mathcal{Y}$ be classically algebraic categories, and let $\mathcal{A} \subseteq \mathcal{X}$ and $\mathcal{B} \subseteq \mathcal{Y}$ be the full subcategories spanned by the compact 1-projective objects. Let $u : \mathcal{Y} \rightarrow \mathcal{X}$ be a functor that preserves limits and sifted colimits. By the adjoint functor theorem, u admits a left adjoint $f : \mathcal{X} \rightarrow \mathcal{Y}$, and by Remark 5.1.4, f carries $\mathcal{A}$ into $\mathcal{B}$. We use the notation

$$\mathcal{P}_{\Sigma}(\mathcal{A}) = \text{Ani}(\mathcal{X}) \begin{matrix} \xrightarrow{\text{Ani}(f)} \\ \xleftarrow{\text{Ani}(u)} \end{matrix} \text{Ani}(\mathcal{Y}) = \mathcal{P}_{\Sigma}(\mathcal{B})$$

for the adjunction induced by the functor $f : \mathcal{A} \rightarrow \mathcal{B}$, and we refer to $\text{Ani}(f)$ and $\text{Ani}(u)$ as the *animations* of f and u , respectively. We note that $\text{Ani}(u)$ again preserves limits and sifted colimits, and it fits into a commutative diagram of ∞ -categories

$$(5.1.12) \quad \begin{array}{ccc} \mathcal{Y} & \xrightarrow{\eta_{\text{Ani}}} & \text{Ani}(\mathcal{Y}) \\ u \downarrow & & \downarrow \text{Ani}(u) \\ \mathcal{X} & \xrightarrow{\eta_{\text{Ani}}} & \text{Ani}(\mathcal{X}). \end{array}$$

Remark 5.1.13. The commutative square (5.1.12) is horizontally left adjointable; this is clear from the description in Remark 5.1.8 of the left adjoint π_0 of η_{Ani} . If $u : \mathcal{Y} \rightarrow \mathcal{X}$ is algebraic (i.e. additionally conservative), then the square is cartesian.

The main observation of this section, which motivates our work in the next section, is the following expression of animation in terms of theorization and algebraization.

Proposition 5.1.14. *Let $u : \mathcal{Y} \rightarrow \mathcal{X}$ be an algebraic functor between classically algebraic categories, and let $\mathcal{A} \subseteq \mathcal{X}$ be the full subcategory of compact 1-project objects. Then the square (5.1.12) exhibits $\text{Ani}(\mathcal{Y})$ as both the $\mathcal{A}$ -theorization and the algebraization of $\mathcal{Y}$ over $\text{Ani}(\mathcal{X})$. (Here we allow ourselves to identify $\mathcal{A}$ with its essential image under η_{Ani} , i.e. under the Yoneda embedding $j_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{P}_{\Sigma}(\mathcal{A})$.)*

Proof. By Remark 2.3.15, algebraicity and $\mathcal{A}$ -theoreticity over $\text{Ani}(\mathcal{X})$ are equivalent, so it suffices to prove the $\mathcal{A}$ -theorization claim. Let $\mathcal{B} \subseteq \mathcal{Y}$ be the full subcategory of compact 1-projective objects and let $\mathcal{B}' \subseteq \mathcal{B}$ denote the essential image of $\mathcal{A}$ under the left adjoint f of u . Identifying $\text{Str}_{\mathcal{A}}(\mathcal{Y})$ with $(\mathcal{B}')^{\text{op}}$ (Example 2.2.16), the unit map $\eta : \text{Ani}(\mathcal{Y}) \rightarrow \text{Mod}_{\text{Ani}(\mathcal{X})}(\text{Str}_{\mathcal{A}}(\mathcal{Y}))$ identifies with the restriction map $\mathcal{P}_{\Sigma}(\mathcal{B}) \rightarrow \mathcal{P}_{\Sigma}(\mathcal{B}')$ (for the identification of the target, see Remark 2.3.13). This is an equivalence by Remark 5.1.7 (applied to $\mathcal{Y}$). $\square$

Example 5.1.15. Let $\mathcal{Y}$ be a locally small, cocomplete category and let $\{B_i\}_{i \in \mathbf{I}}$ be a detecting set of compact 1-projective objects of $\mathcal{Y}$. Then we may regard $\mathcal{Y}$ and its animation $\text{Ani}(\mathcal{Y})$ as ∞ -categories over $\text{Spc}^{\mathbf{I}} \simeq \text{Ani}(\text{Set}^{\mathbf{I}})$ via the functors corepresented by this set of objects. The canonical embedding $\eta_{\text{Ani}} : \mathcal{Y} \rightarrow \text{Ani}(\mathcal{Y})$ exhibits the animation of $\mathcal{Y}$ as the $\Sigma^{\mathbf{I}}$ -theorization/algebraization of $\mathcal{Y}$ over $\text{Spc}^{\mathbf{I}}$.

Remark 5.1.16. We can alternatively phrase Remark 5.1.13 and Proposition 5.1.14 using notions from Ch. 3. Namely, the identity map of $\mathcal{T} := \text{Str}_{\mathcal{A}}(\mathcal{Y})$ defines a map $\mu : (\text{Ani}(\mathcal{X})|_{\mathcal{A}}, \mathcal{T}) \rightarrow (\mathcal{X}|_{\mathcal{A}}, \mathcal{T})$ in Thy lying over the map $\pi_0 : \text{Ani}(\mathcal{X})|_{\mathcal{A}} \rightarrow \mathcal{X}|_{\mathcal{A}}$ in $\text{Prd}^{\mathbf{L}}$. Proposition 5.1.14 is the observation that the square (5.1.12) is the image of μ under Mod (see Remark 3.2.4). And Remark 5.1.13 is the observation that μ is model adjointable over π_0 (Definition 3.3.4).

§5.2. Bianimation. The setup now is as follows.

Notation 5.2.1. Throughout this section (as well as the next two), let $\mathcal{X}$ be a category that is abelian and classically algebraic (Definition 5.1.1). Let $\mathcal{A} \subseteq \mathcal{X}$ be the full subcategory of compact 1-projective objects,

and let $\mathfrak{X}_{\geq 0}$ denote the animation $\text{Ani}(\mathcal{X}) = \mathcal{P}_{\Sigma}(\mathcal{A})$ (Definition 5.1.6). We'll sometimes identify $\mathcal{X}$ (and in particular $\mathcal{A}$) with its essential image under the embedding $\eta_{\text{Ani}} : \mathcal{X} \hookrightarrow \mathfrak{X}_{\geq 0}$.

Let $\mathfrak{X}$ denote the stabilization

$$\text{Spt}(\mathfrak{X}_{\geq 0}) \simeq \lim (\cdots \xrightarrow{\Omega} \mathfrak{X}_{\geq 0} \xrightarrow{\Omega} \mathfrak{X}_{\geq 0} \xrightarrow{\Omega} \mathfrak{X}_{\geq 0}),$$

which we may identify with the full subcategory of $\text{Fun}(\mathcal{A}^{\text{op}}, \text{Spt})$ spanned by the finite product preserving functors. By [29, Remark C.1.5.9], we may then identify $\mathfrak{X}_{\geq 0}$ with the full subcategory of $\mathfrak{X}$ spanned by those such functors whose values are connective spectra, and this defines the connective part of a right complete t-structure on $\mathfrak{X}$ (justifying our notation).

Finally, let $\mathfrak{A} \subseteq \mathfrak{X}$ be the full subcategory generated by $\mathcal{A}$ under desuspensions, so that $\mathfrak{X}|\mathfrak{A}$ is a presented ∞ -category of the type studied in Ch. 4. The full subcategory $\mathfrak{A}_0 := \mathcal{A} \subseteq \mathfrak{A}$ furthermore puts us in the situation studied in §4.4 (see Notation 4.4.2).

Remark 5.2.2. Recall from Example 5.1.9 that $\mathfrak{X}_{\geq 0} \simeq \text{D}(\mathcal{X})_{\geq 0}$. This extends to a t-exact equivalence $\mathfrak{X} \simeq \text{D}(\mathcal{X})$, since the t-structure on $\text{D}(\mathcal{X})$ is also right complete [26, Proposition 1.3.5.21].

Example 5.2.3. Suppose that $\mathcal{X} = \text{Mod}_R^{\heartsuit} = \text{Mod}_R(\text{Ab})$, the classical category of left R -modules for R a ring. Then we also have a canonical t-exact equivalence $\mathfrak{X} \simeq \text{Mod}_R$, where $\text{Mod}_R = \text{Mod}_R(\text{Spt})$ is the ∞ -category of left R -module spectra. More generally, if $\mathcal{X} = (\text{Mod}_R^{\heartsuit})^I$ for a set I , then $\mathfrak{X} \simeq \text{Mod}_R^I$.

We saw in the previous section that, for algebraic categories over $\mathcal{X}$, animation could be recast as $\mathcal{A}$ -theorization or algebraization over $\mathfrak{X}_{\geq 0}$. We will now make an analogous construction over $\mathfrak{X}$, by working with $\mathfrak{X}|\mathfrak{A}$ -theories.

Notation 5.2.4. We write $\iota : \mathfrak{X}_{\geq 0} \hookrightarrow \mathfrak{X}$ for the inclusion functor and we set $\delta := \iota \circ \eta_{\text{Ani}} : \mathcal{X} \hookrightarrow \mathfrak{X}$.

Construction 5.2.5. Let $u : \mathcal{Y} \rightarrow \mathcal{X}$ be an algebraic functor. The composition $\delta u : \mathcal{Y} \rightarrow \mathfrak{X}$ is accessible, hence $\mathfrak{A}$ -small, hence $\mathfrak{A}$ -theorizable (Example 2.2.17). We thus have an associated $\mathfrak{X}|\mathfrak{A}$ -theory $\text{Str}_{\mathfrak{A}}(\mathcal{Y}) \subseteq \mathcal{P}^{\vee}(\mathcal{Y})$ and $\mathfrak{A}$ -theorization

$$\eta : \mathcal{Y} \rightarrow \mathcal{Y}^{\mathfrak{A}\text{-th}/\mathfrak{X}} := \text{Mod}_{\mathfrak{X}}(\text{Str}_{\mathfrak{A}}(\mathcal{Y}))$$

over $\mathfrak{X}$ (Remark 4.2.3).

Remark 5.2.6. Suppose that $\mathcal{X} = \text{Ab}^I$ for a set I , so that we have $\mathfrak{X} \simeq \text{Mod}_{\mathbb{Z}}^I$ (Example 5.2.3). Let us unravel Construction 5.2.5 into more explicit terms in this case:

1. The ∞ -category $\mathfrak{T} := \text{Str}_{\mathfrak{A}}(\mathcal{Y})$ is the full subcategory of $\text{Fun}(\mathcal{Y}, \text{Spc})$ generated under finite products by the functors $B^n u_i : \mathcal{Y} \rightarrow \text{Spc}$ for $i \in I$ and $n \in \mathbb{Z}_{\geq 0}$. Here B^n denotes the n -fold classifying space construction (see Remark 4.4.7 and Example 4.4.8). Note that, for $n = 0$, the functors $B^n u_i = u_i$ are corepresentable, namely by $f(\mathbb{Z}(-i))$, where $f : \text{Ab}^I \rightarrow \mathcal{Y}$ is the left adjoint of u and $\mathbb{Z}(-i)$ is as in Convention 0.3.12; more generally, a finite product of these functors $u_{i_1} \times \cdots \times u_{i_k}$ is corepresented by $f(\mathbb{Z}(-i_1) \oplus \cdots \oplus \mathbb{Z}(-i_k)) \simeq f(\mathbb{Z}(-i_1)) \sqcup \cdots \sqcup f(\mathbb{Z}(-i_k))$.
2. The ∞ -category $\mathfrak{Y} := \mathcal{Y}^{\mathfrak{A}\text{-th}/\mathfrak{X}}$ is the full subcategory of $\text{Fun}(\mathfrak{T}, \text{Spc})$ spanned by those functors $M : \mathfrak{T} \rightarrow \text{Spc}$ that preserve finite products and the canonical pullback diagram

$$\begin{array}{ccc} B^n u_i & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & B^{n+1} u_i \end{array}$$

for each $i \in I$ and $n \in \mathbb{Z}_{\geq 0}$.

3. The functor $\eta : \mathcal{Y} \rightarrow \mathfrak{Y}$ carries an object $Y \in \mathcal{Y}$ to the evaluation functor at Y . We also have the Yoneda embedding $j_{\mathfrak{T}}^{\vee} : \mathfrak{T}^{\text{op}} \rightarrow \mathfrak{Y}$ and the structure map $\mathfrak{u} : \mathfrak{Y} \rightarrow \mathfrak{X} \simeq \text{Mod}_{\mathbb{Z}}^I$. The structure map is described by the formula

$$\Omega^{\infty-n} \mathfrak{u}(M)_i \simeq M(B^n u_i)$$

for $i \in \mathbf{I}$ and $n \in \mathbb{Z}_{\geq 0}$; equivalently, $\mathbf{u}$ has a left adjoint $\mathbf{f} : \text{Mod}_{\mathbb{Z}}^{\mathbf{I}} \rightarrow \mathcal{Y}$ carrying $\mathbb{Z}(-i)[-n]$ to $\mathbf{j}_{\mathfrak{X}}^{\vee}(\mathbf{B}^n u_i)$.

The composition $\mathbf{u} \circ \mathbf{j}_{\mathfrak{X}}^{\vee} : \mathfrak{T}^{\text{op}} \rightarrow \mathfrak{X}$ identifies with the restriction of the functor $\Gamma_{\delta u} : \mathcal{P}^{\vee}(\mathcal{Y})^{\text{op}} \rightarrow \text{Mod}_{\mathbb{Z}}^{\mathbf{I}}$, i.e. the right Kan extension of $\delta u : \mathcal{Y} \rightarrow \text{Mod}_{\mathbb{Z}}^{\mathbf{I}}$ along the Yoneda embedding $\mathbf{j}_{\mathcal{Y}}^{\vee} : \mathcal{Y} \rightarrow \mathcal{P}^{\vee}(\mathcal{Y})^{\text{op}}$ (Proposition 2.2.27). In particular, for any $i_1, \dots, i_k \in \mathbf{I}$, we have

$$\begin{aligned} \mathbf{uf}(\mathbb{Z}(-i_1) \oplus \dots \oplus \mathbb{Z}(-i_k)) &\simeq \mathbf{u}(\mathbf{j}_{\mathfrak{X}}^{\vee}(u_{i_1} \times \dots \times u_{i_k})) \\ &\simeq \Gamma_{\delta u}(u_{i_1} \times \dots \times u_{i_k}) \\ &\simeq \delta u f(\mathbb{Z}(-i_1) \oplus \dots \oplus \mathbb{Z}(-i_k)); \end{aligned}$$

and for $i \in \mathbf{I}$ and $n \in \mathbb{Z}_{\geq 1}$, we have

$$\begin{aligned} \mathbf{uf}(\mathbb{Z}(-i)[-n]) &\simeq \mathbf{u}(\mathbf{j}_{\mathfrak{X}}^{\vee}(\mathbf{B}^n u_i)) \simeq \Gamma_{\delta u}(\mathbf{B}^n u_i) \\ &\simeq \lim_{\Delta} \Gamma_{\delta u}(\text{bar}_{\mathbf{B}^{n-1} u_i}^{\times}(1, 1) \bullet) \\ &\simeq \lim_{\Delta} \mathbf{uf}(\text{cobar}_{\mathbb{Z}(-i)[-n+1]}(0, 0)^{\bullet}). \end{aligned}$$

Remark 5.2.7. In Remark 5.2.6, the formulas for the monad $\mathbf{t} := \mathbf{uf} : \mathfrak{X} \rightarrow \mathfrak{X}$ can be understood in terms of the following relations to the monad $t := uf : \mathcal{X} \rightarrow \mathcal{X}$:

1. the restriction $\mathbf{t}|_{\mathcal{A}}$ identifies with $t|_{\mathcal{A}}$;
2. the restriction $\mathbf{t}|_{\mathfrak{A}}$ is right Kan extended along the inclusion $\mathcal{A} \hookrightarrow \mathfrak{A}$.

These assertions hold for any $\mathfrak{X}$, not just $\mathfrak{X} = \text{Ab}^{\mathbf{I}}$; the second is a special case of Example 4.4.8, and by Proposition 4.4.12 is equivalent to the condition that $\mathbf{t}|_{\mathfrak{A}}$ preserve cobar constructions.

Unlike the situation for animation described in the previous section, it isn't clear that the theorization of Construction 5.2.5 is algebraic over $\mathfrak{X}$; only when that is satisfied will we use the terminology that titles this chapter and section.²

Definition 5.2.8. Let $\mathcal{Y}$ be an algebraic category over $\mathcal{X}$. We say that $\mathcal{Y}$ is *bianimable* over $\mathcal{X}$ if the $\mathfrak{A}$ -theorization $\mathcal{Y}^{\mathfrak{A}\text{-th}/\mathfrak{X}}$ of Construction 5.2.5 is algebraic over $\mathfrak{X}$, i.e. the structure map $\mathcal{Y}^{\mathfrak{A}\text{-th}/\mathfrak{X}} \rightarrow \mathfrak{X}$ preserves sifted colimits, or equivalently the associated monad $\mathbf{t} : \mathfrak{X} \rightarrow \mathfrak{X}$ preserves sifted colimits. If this holds, we set

$$\text{bAni}(\mathcal{Y}/\mathcal{X}) := \mathcal{Y}^{\mathfrak{A}\text{-th}/\mathfrak{X}}$$

and alternatively refer to this ∞ -category over $\mathfrak{X}$ as the *bianimation* of $\mathcal{Y}$ over $\mathcal{X}$; we may sometimes, for clarity, write $\eta_{\text{bAni}} : \mathcal{Y} \rightarrow \text{bAni}(\mathcal{Y}/\mathcal{X})$ for the $\mathfrak{A}$ -theorization functor denoted just by η in Construction 5.2.5.

Remark 5.2.9. In Definition 5.2.8, note that $\mathbf{t}$ preserving sifted colimits implies that it is left Kan extended from $\mathfrak{A}$ (Remark 2.3.7). On the other hand, the restriction $\mathbf{t}|_{\mathfrak{A}}$ is right Kan extension from $\mathcal{A}$ (Remark 5.2.7). Thus, bianimation may be understood as a reformulation of the procedure of *right-left extension* of monads on $\mathcal{X}$ to monads on $\mathfrak{X}$, due originally to Brantner–Mathew [12, §3.2] and further developed by Brantner–Campos–Nuiten [10, §2.2] and Brantner–Magidston–Nuiten [11, §A.1]. We'll return to this point at the end of the section (Remark 5.2.23), after discussing the construction from our theoretical point of view.

Before saying anything more about it, let us show that, when it is defined, bianimation furnishes an answer to Question 5.0.1, which moreover is optimal in a certain sense.

Theorem 5.2.10. *Let $\mathcal{Y}$ be a bianimable category over $\mathcal{X}$. Then the following statements hold:*

²This is a choice made out of (a certain degree of) conservativity and ignorance on my part: I don't know an example where it is not algebraic, and if there is one, I don't know whether the theorization would or would not be a good analogue of animation to consider (though see Remark 5.2.11 below).

1. *There is a canonical cartesian square*

$$\begin{array}{ccc} \mathrm{Ani}(\mathcal{Y}) & \xhookrightarrow{\theta} & \mathrm{bAni}(\mathcal{Y}/\mathcal{X}) \\ \mathrm{Ani}(u) \downarrow & & \downarrow u \\ \mathfrak{X}_{\geq 0} & \xhookrightarrow{\iota} & \mathfrak{X}, \end{array}$$

where $u : \mathcal{Y} \rightarrow \mathcal{X}$ and $u : \mathrm{bAni}(\mathcal{Y}/\mathcal{X}) \rightarrow \mathfrak{X}$ denote the structure maps, and θ is characterized by being colimit preserving and carrying a canonical identification $\theta \circ \eta_{\mathrm{Ani}} \simeq \eta_{\mathrm{bAni}} : \mathcal{Y} \rightarrow \mathrm{bAni}(\mathcal{Y}/\mathcal{X})$. Furthermore, θ exhibits $\mathrm{bAni}(\mathcal{Y}/\mathcal{X})$ as the $\mathfrak{A}$ -theorization of $\mathrm{Ani}(\mathcal{Y})$ over $\mathfrak{X}$, and the square is horizontally right adjointable.

2. *The functors $\eta_{\mathrm{bAni}} : \mathcal{Y} \rightarrow \mathrm{bAni}(\mathcal{Y}/\mathcal{X})$ and $\theta : \mathrm{Ani}(\mathcal{Y}) \rightarrow \mathrm{bAni}(\mathcal{Y}/\mathcal{X})$ exhibit $\mathrm{bAni}(\mathcal{Y}/\mathcal{X})$ as the algebraization of both $\mathcal{Y}$ and $\mathrm{Ani}(\mathcal{Y})$ over $\mathfrak{X}$.*

Proof. Set $\mathfrak{T} := \mathrm{Str}_{\mathfrak{A}}(\mathcal{Y})$, as defined in Construction 5.2.5, so that $\mathrm{bAni}(\mathcal{Y}/\mathcal{X}) = \mathrm{Mod}_{\mathfrak{X}}(\mathfrak{T})$. Let $\mathcal{T} := \mathfrak{T}_0$ be the restriction along ι of $\mathfrak{T}$, an $\mathfrak{X}_{\geq 0}|\mathcal{A}$ -theory (Notation 4.4.6). We then have

$$\mathrm{Mod}_{\mathfrak{X}_{\geq 0}}(\mathcal{T}) = \mathcal{P}_{\Sigma}^{\vee}(\mathcal{T}) = \mathrm{Ani}(\mathcal{Y}),$$

(the first equality being explained in Remark 4.4.5 and the second equality following from the definition of the animation $\mathrm{Ani}(\mathcal{Y})$), and it is clear that $\mathfrak{T}$ is ι -consistent. Proposition 4.4.9.1 now gives us the cartesian square of 1, as well as its horizontal right adjointability: the functor θ is defined by left Kan extension along the inclusion $\mathcal{T} \hookrightarrow \mathfrak{T}$, with right adjoint $\tau : \mathrm{bAni}(\mathcal{Y}/\mathcal{X}) \rightarrow \mathrm{Ani}(\mathcal{Y})$ given by restriction along this inclusion; and the claimed identification $\theta \circ \eta_{\mathrm{Ani}} \simeq \eta_{\mathrm{bAni}}$ is equivalent to an identification $\eta_{\mathrm{Ani}} \simeq \tau \circ \eta_{\mathrm{bAni}}$, which is found immediately after unravelling the definitions. That θ exhibits $\mathrm{bAni}(\mathcal{Y}/\mathcal{X})$ as the $\mathfrak{A}$ -theorization of $\mathrm{Ani}(\mathcal{Y})$ over $\mathfrak{X}$ follows from Proposition 4.4.9.2, since $\mathfrak{T}$ is ι -extended by Example 4.4.8. This completes the proof of 1. Assertion 2 then follows from Remark 4.2.10, since u is algebraic by definition of bianimability. $\square$

Remark 5.2.11. The statement and proof of Theorem 5.2.10.1 go through without the bianimability hypothesis on $\mathcal{Y}$ (except that, to follow the conventions set above, we should revert to the notation $\mathcal{Y}^{\mathfrak{A}\text{-th}/\mathfrak{X}}$ instead of writing $\mathrm{bAni}(\mathcal{Y}/\mathcal{X})$).

Regarding the condition of bianimability, we recall the following reductions from the general results of Ch. 4.

Proposition 5.2.12. *Let $u : \mathcal{Y} \rightarrow \mathcal{X}$ be an algebraic functor. Then the following conditions are equivalent:*

1. *$\mathcal{Y}$ is bianimable over $\mathcal{X}$;*
2. *the functor $\delta u : \mathcal{Y} \rightarrow \mathfrak{X}$ satisfies the $\mathfrak{A}$ -theoretical Eilenberg–Moore bar formula (Definition 4.2.5);*
3. *writing $t : \mathcal{X} \rightarrow \mathcal{X}$ for the monad associated to u and $t|_{\mathfrak{A}} : \mathfrak{A} \rightarrow \mathfrak{X}$ for the right Kan extension of $t|_{\mathcal{A}}$ along the inclusion $\mathcal{A} \hookrightarrow \mathfrak{A}$, we have that $t|_{\mathfrak{A}}$ preserves bar constructions (Definition 4.1.13).*

Proof. The equivalence $1 \Leftrightarrow 2$ is a special case of Theorem 4.2.9.3, and the equivalence $2 \Leftrightarrow 3$ follows from unwinding definitions. The composite equivalence $1 \Leftrightarrow 3$ can equivalently be understood as the combination of Theorem 2.3.11, Proposition 4.1.15, and Remark 5.2.7. $\square$

Remark 5.2.13. The bar construction condition of Proposition 5.2.12 may not seem the most inviting for direct verification. Let us describe a strategy for establishing it. We first note the following: suppose that we have a commutative diagram of ∞ -categories

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{v} & \mathfrak{Z} \\ u \downarrow & & \downarrow \mathfrak{w} \\ \mathcal{X} & \xrightarrow{\delta} & \mathfrak{X} \end{array}$$

in which $\mathfrak{w}$ is algebraic; then δu satisfies the $\mathfrak{A}$ -theoretical Eilenberg–Moore bar formula (i.e. $\mathcal{Y}$ is bianimable over $\mathcal{X}$) if and only if v satisfies the $\mathfrak{A}$ -theoretical Eilenberg–Moore formula. Now, it may be possible to find such a commutative diagram such that v satisfies the $\mathfrak{A}$ -theoretical Künneth formula, i.e.

$\Gamma_v : \text{Str}_{\mathfrak{A}}(\mathcal{Y})^{\text{op}} \rightarrow \mathfrak{Z}$ preserves finite coproducts (though this is true of δu itself if and only if $\mathcal{Y}$ is abelian). In this case, the $\mathfrak{A}$ -theoretical Eilenberg–Moore bar formula for v is equivalent to the suspension form of the $\mathfrak{A}$ -theoretical Eilenberg–Moore formula, namely that, for each $A \in \mathfrak{A}$, the canonical map

$$\Gamma_v(1) \sqcup_{\Gamma_v(\chi(A[-1]))} \Gamma_v(1) \rightarrow \Gamma_v(\chi(A))$$

in $\mathfrak{Z}$ is an equivalence (where $\chi : \mathfrak{A}^{\text{op}} \rightarrow \text{Str}_{\mathfrak{A}}(\mathcal{Y})$ is the structure map). See Remark 5.4.6 for one illustration of this strategy.

We now make several notes about the invariance and functoriality properties of bianimation.

Remark 5.2.14. As indicated by our notation, it’s important to keep in mind that Construction 5.2.5 and Definition 5.2.8 depend on $\mathcal{Y}$ not just as a category, but as a category over $\mathfrak{X}$. This is in contrast with the definition of animation reviewed in the previous section. From the theoretical point of view on animation (Proposition 5.1.14), its invariance ultimately comes from the fact that any set of compact 1-projective generators of $\mathcal{Y}$ generates all of its compact 1-projective objects under finite coproducts and retracts (Remark 2.3.19). An analogous phenomenon does not seem to hold generally in the setting of Construction 5.2.5 (although it does partially; see Remark 5.2.16).

For example, consider the category $\mathcal{Y} = \text{Ab} \times_{\text{Set}} \text{Ab}$. Each of the two projection functors $u_0 : \mathcal{Y} \rightarrow \text{Ab}$ and $u_1 : \mathcal{Y} \rightarrow \text{Ab}$ is algebraic. However, the functor $Bu_1 : \mathcal{Y} \rightarrow \text{Spc}$ is not generated under finite products and retracts by the functors $B^n u_0$ for $n \in \mathbb{Z}_{\geq 0}$. To see this, let Y_0 be a free abelian group of rank 1 and let Y_1 be a free abelian group of countably infinite rank. We may choose a bijection between the two underlying sets (both being countably infinite) and thereby define an object $Y \in \mathcal{Y}$ with $u_0(Y) = Y_0$ and $u_1(Y) = Y_1$. Now we observe that the space BY_1 is not a retract of a finite product of the spaces $B^n Y_0$ for $n \in \mathbb{Z}_{\geq 0}$: if it were, then, by passing to fundamental groups, we would find that the group Y_1 is a retract of a finite product of copies of Y_0 , which is not true.

On the other hand, there may be a more subtle mechanism for further invariance of bianimation than is exhibited here.

Remark 5.2.15. Any of the functorialities of theorization studied in Chs. 2 and 3 may be contemplated in the context of this section. For now, let us just record an analogue of Construction 5.1.11. Suppose given a commutative square of categories

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{\sigma} & \mathcal{Y}' \\ u \downarrow & & \downarrow u' \\ \mathfrak{X} & \xrightarrow{\rho} & \mathfrak{X}' \end{array}$$

in which: $\mathfrak{X}'$ is also abelian and classically algebraic; u and u' are algebraic; and ρ (and hence σ) preserves limits and sifted colimits. Let $\mathcal{A}' \subseteq \mathfrak{X}'$ be the full subcategory spanned by the compact 1-projective objects, let $\mathfrak{X}' := \text{Spt}(\text{Ani}(\mathfrak{X}'))$, and let $\mathfrak{A}' \subseteq \mathfrak{X}'$ be the full subcategory generated by $\mathcal{A}'$ under desuspensions. As in Construction 5.1.11, the functor ρ has a left adjoint $\lambda : \mathfrak{X}' \rightarrow \mathfrak{X}$ carrying $\mathcal{A}'$ into $\mathcal{A}$, inducing an animated adjunction $\text{Ani}(\lambda) : \text{Ani}(\mathfrak{X}') \rightleftarrows \text{Ani}(\mathfrak{X}) : \text{Ani}(\rho)$. Stabilizing then gives an adjunction $\mathfrak{l} : \mathfrak{X}' \rightleftarrows \mathfrak{X} : \mathfrak{r}$, where $\mathfrak{l}$ carries $\mathfrak{A}'$ into $\mathfrak{A}$ and $\mathfrak{r}$ fits into a commutative square

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{\rho} & \mathfrak{X}' \\ \delta \downarrow & & \downarrow \delta' \\ \mathfrak{X} & \xrightarrow{\mathfrak{r}} & \mathfrak{X}' \end{array}$$

(δ' being defined analogously to δ). Pasting the two preceding squares together, we get an induced functor on theorizations $\mathcal{Y}^{\mathfrak{A}\text{-th}/\mathfrak{X}} \rightarrow (\mathcal{Y}')^{\mathfrak{A}'\text{-th}/\mathfrak{X}'}$ (Remark 3.2.9). If $\mathcal{Y}$ is bianimable over $\mathfrak{X}$ and $\mathcal{Y}'$ is bianimable over $\mathfrak{X}'$, then we may alternatively denote this functor by

$$\text{bAni}(\sigma/\rho) : \text{bAni}(\mathcal{Y}/\mathfrak{X}) \rightarrow \text{bAni}(\mathcal{Y}'/\mathfrak{X}')$$

and refer to it as the *bianimation* of σ over ρ .

Remark 5.2.16. Let $\mathfrak{X}'$ be a stable, presentable ∞ -category equipped with an essentially small, detecting subcategory of compact objects $\mathfrak{A}' \subseteq \mathfrak{X}'$ that is closed under desuspensions. Let $\mathfrak{r} : \mathfrak{X} \rightarrow \mathfrak{X}'$ be functor that is algebraic (equivalently $\mathfrak{A}'$ -theoretical or monadic and colimit preserving, by Proposition 4.3.1), let $\mathfrak{l} : \mathfrak{X} \rightarrow \mathfrak{X}'$ be its left adjoint, and assume that the essential image of $\mathfrak{A}'$ under $\mathfrak{l}$ is contained in $\mathfrak{A}$ and generates $\mathfrak{A}$ under retracts. Then, in the situation of Construction 5.2.5, we have an equivalence

$$\mathcal{Y}^{\mathfrak{A}\text{-th}/\mathfrak{X}} \xrightarrow{\sim} \mathcal{Y}^{\mathfrak{A}'\text{-th}/\mathfrak{X}'}$$

(where, on the right side, we regard $\mathcal{Y}$ as a category over $\mathfrak{X}'$ by composition with $\mathfrak{r}$), induced by the inclusion $\text{Str}_{\mathfrak{A}'}(\mathcal{Y}) \hookrightarrow \text{Str}_{\mathfrak{A}}(\mathcal{Y})$. This follows from combining Remark 3.2.13 and Proposition 3.2.14. As a consequence, if $\mathcal{Y}$ is bianimable over $\mathfrak{X}$, then η_{bAni} also exhibits $\text{bAni}(\mathcal{Y}/\mathfrak{X})$ as the algebraization of $\mathcal{Y}$ over $\mathfrak{X}'$. These comments also apply with $\text{Ani}(\mathcal{Y})$ in place of $\mathcal{Y}$ (see Theorem 5.2.10).

Example 5.2.17. In the situation of Remark 5.2.15, suppose that $\mathcal{Y} = \mathcal{Y}'$ and $\sigma = \text{id}$, and suppose that ρ is algebraic. Then the induced adjunction $\mathfrak{l} : \mathfrak{X}' \rightleftarrows \mathfrak{X} : \mathfrak{r}$ is of the type postulated in Remark 5.2.16. It follows that $\mathcal{Y}$ is bianimable over $\mathfrak{X}$ if and only if it is so over $\mathfrak{X}'$, and when this condition holds, we have an equivalence

$$\text{bAni}(\text{id}/\rho) : \text{bAni}(\mathcal{Y}/\mathfrak{X}) \xrightarrow{\sim} \text{bAni}(\mathcal{Y}/\mathfrak{X}').$$

Let us highlight a special case of this situation that may be arranged for any $\mathfrak{X}$: choosing any detecting set of compact 1-projective objects $\{A_i\}_{i \in I}$ in $\mathcal{A} \subseteq \mathfrak{X}$, we may take $\mathfrak{X}' = \text{Ab}^I$ and $\rho : \mathfrak{X} \rightarrow \mathfrak{X}'$ to be the functor corepresented by this set of objects. In this way, bianimation may always be understood in terms of the concrete description of Remark 5.2.6.

Example 5.2.18. Example 5.2.17 shows that, in a certain sense, bianimation reduces to the case where $\mathfrak{X} = \text{Ab}^I$ for some set I , so that $\mathfrak{X} \simeq \text{Mod}_{\mathbb{Z}}^I$. Supposing that we are in this case, Remark 5.2.16 may be applied again by taking $\mathfrak{X}' := \text{Spt}^I$ and $\mathfrak{A}' := \mathcal{A}^I$ (Example 4.0.2), and choosing $\mathfrak{r} : \mathfrak{X} \rightarrow \mathfrak{X}'$ to be the forgetful functor. This gives us a further identification

$$\mathcal{Y}^{\mathfrak{A}\text{-th}/\mathfrak{X}} \xrightarrow{\sim} \mathcal{Y}^{\mathcal{A}^I\text{-th}/\text{Spt}^I},$$

and if $\mathcal{Y}$ is bianimable over $\mathfrak{X}$ then, $\text{bAni}(\mathcal{Y}/\mathfrak{X})$ is the algebraization of $\mathcal{Y}$ and of $\text{Ani}(\mathcal{Y})$ over Spt^I .

Remark 5.2.19. The maneuver of Remark 5.2.16 and Examples 5.2.17 and 5.2.18 can also be applied in the opposite direction, in a similar spirit to Remark 5.2.13. Suppose that we have a commutative diagram of ∞ -categories

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{v} & \mathfrak{Z} \\ u \downarrow & & \downarrow \mathfrak{w} \\ \mathfrak{X} & \xrightarrow{\delta} & \mathfrak{X} \end{array}$$

in which $\mathfrak{w}$ is $\mathfrak{A}$ -theoretical and where u is algebraic as above. Then, by the universal property of $\mathfrak{A}$ -theorization, the functor v factors uniquely through a functor $v' : \mathcal{Y}^{\mathfrak{A}\text{-th}/\mathfrak{X}} \rightarrow \mathfrak{Z}$ over $\mathfrak{X}$. In fact, this exhibits $\mathcal{Y}^{\mathfrak{A}\text{-th}/\mathfrak{X}}$ as the $\mathfrak{A}_{\mathfrak{Z}}$ -theorization of $\mathcal{Y}$ over $\mathfrak{Z}$, where $\mathfrak{A}_{\mathfrak{Z}}$ is the essential image of $\mathfrak{A}$ under the left adjoint of $\mathfrak{w}$ (Remark 3.2.13).

To conclude this section, we observe that bianimation establishes an equivalence between bianimable categories over $\mathfrak{X}$ and a certain class of algebraic categories over $\mathfrak{X}$.

Definition 5.2.20. We say that an endofunctor $\mathfrak{t} : \mathfrak{X} \rightarrow \mathfrak{X}$ is *bianimated* if it preserves sifted colimits, it carries $\mathcal{A}$ into $\mathfrak{X}$, and its restriction $\mathfrak{t}|_{\mathfrak{A}}$ is right Kan extended along the inclusion $\mathcal{A} \hookrightarrow \mathfrak{A}$ (equivalently preserves cobar constructions, by Proposition 4.4.12).

We say that an algebraic ∞ -category $\mathfrak{Y}$ over $\mathfrak{X}$ is *bianimated* if its associated monad $\mathfrak{t} : \mathfrak{X} \rightarrow \mathfrak{X}$, which preserves sifted colimits by algebraicity, is furthermore bianimated.

Remark 5.2.21. Definition 5.2.20 is related to the notions of ι -consistency and ι -extendedness of $\mathfrak{X}|\mathfrak{A}$ -theories (Remark 4.4.7), as follows. Let $\mathfrak{Y}$ be an algebraic ∞ -category over $\mathfrak{X}$, let $\mathfrak{T} := \text{Str}_{\mathfrak{A}}(\mathfrak{Y})$ be the

associated $\mathfrak{X}|\mathfrak{A}$ -theory, and let $\mathfrak{t} : \mathfrak{X} \rightarrow \mathfrak{X}$ be the associated monad. The condition that $\mathfrak{t}$ carry $\mathcal{A}$ into $\mathfrak{X}$ means in particular that $\mathfrak{T}$ is ι -consistent (since $\mathfrak{X} \subseteq \mathfrak{X}_{\geq 0}$), and the condition that $\mathfrak{t}|_{\mathfrak{A}}$ be right Kan extended from $\mathcal{A}$ is equivalent to $\mathfrak{T}$ being ι -extended.

Proposition 5.2.22. *Let $\text{Cat}_{/\mathfrak{X}}^{\text{banibl}} \subseteq \text{Cat}_{/\mathfrak{X}}$ and $\text{Cat}_{/\mathfrak{X}}^{\text{banid}} \subseteq \text{Cat}_{/\mathfrak{X}}$ be the full subcategories spanned by the bianimable categories over $\mathfrak{X}$ (Definition 5.2.8) and bianimated ∞ -categories over $\mathfrak{X}$ (Definition 5.2.20), respectively. Then banimation and pullback along $\delta : \mathfrak{X} \hookrightarrow \mathfrak{X}$ define an equivalence of ∞ -categories*

$$\text{bAni}(-/\mathfrak{X}) : \text{Cat}_{/\mathfrak{X}}^{\text{banibl}} \simeq \text{Cat}_{/\mathfrak{X}}^{\text{banid}} : \delta^*.$$

Proof. Let $\mathcal{Y}$ be a bianimable category over $\mathfrak{X}$. Then $\text{bAni}(\mathcal{Y}/\mathfrak{X})$ is a bianimated ∞ -category over $\mathfrak{X}$ by definition of bianimability and Remark 5.2.7. Furthermore, by its construction, we have a natural commutative square

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{\eta_{\text{bAni}}} & \text{bAni}(\mathcal{Y}/\mathfrak{X}) \\ u \downarrow & & \downarrow u \\ \mathfrak{X} & \xrightarrow{\delta} & \mathfrak{X}, \end{array}$$

where u and u are the structure maps. Combining Remark 5.1.13 and Theorem 5.2.10.1 gives that this square is cartesian, which establishes the equivalence $\delta^* \circ \text{bAni}(-/\mathfrak{X}) \simeq \text{id} : \text{Cat}_{/\mathfrak{X}}^{\text{banibl}} \rightarrow \text{Cat}_{/\mathfrak{X}}^{\text{banibl}}$.

Now let $\mathfrak{Y}$ be a bianimated ∞ -category over $\mathfrak{X}$. Let $u : \mathfrak{Y} \rightarrow \mathfrak{X}$ denote the structure map and $\mathfrak{t} : \mathfrak{X} \rightarrow \mathfrak{X}$ the associated monad. We form cartesian squares

$$\begin{array}{ccccc} \mathcal{Y} & \xrightarrow{\eta} & \mathfrak{Y}_{\geq 0} & \xrightarrow{\theta} & \mathfrak{Y} \\ u \downarrow & & \downarrow u_{\geq 0} & & \downarrow u \\ \mathfrak{X} & \xrightarrow{\eta_{\text{Ani}}} & \mathfrak{X}_{\geq 0} & \xrightarrow{\iota} & \mathfrak{X}. \end{array}$$

Since u is algebraic, it is $\mathfrak{A}$ -theoretical (Theorem 2.3.10). By Remark 5.2.21, the structural $\mathfrak{X}|\mathfrak{A}$ -theory $\mathfrak{T} := \text{Str}_{\mathfrak{A}}(\mathfrak{Y})$ is ι -consistent. Letting $\mathcal{T} := \mathfrak{T}_0$ be the restricted $\mathfrak{X}_{\geq 0}|\mathcal{A}$ -theory, Proposition 4.4.9 identifies $\mathfrak{Y}_{\geq 0}$ with $\text{Mod}_{\mathfrak{X}_{\geq 0}}(\mathcal{T}) = \mathcal{P}_{\Sigma}^{\vee}(\mathcal{T})$. It follows that $\mathcal{Y}$ identifies with $\text{Mod}_{\mathfrak{X}}(\mathcal{T})$ (the full subcategory of 0-truncated objects in $\mathcal{P}_{\Sigma}^{\vee}(\mathcal{T})$). This shows that $\mathcal{Y}$ is algebraic over $\mathfrak{X}$.

By the universal property of $\mathfrak{A}$ -theorization, the functor $v := \theta \circ \eta$ factors uniquely through a functor $v' : \mathcal{Y}^{\mathfrak{A}\text{-th}/\mathfrak{X}} \rightarrow \mathfrak{Y}$ over $\mathfrak{X}$. The functor v' is an equivalence: this follows from the criterion of Remark 4.4.10, using that $\mathfrak{T}$ is ι -extended (Remark 5.2.21). This shows that $\mathcal{Y}$ is bianimable over $\mathfrak{X}$ and establishes the equivalence $\text{bAni}(-/\mathfrak{X}) \circ \delta^* \simeq \text{id} : \text{Cat}_{/\mathfrak{X}}^{\text{banid}} \rightarrow \text{Cat}_{/\mathfrak{X}}^{\text{banid}}$. $\square$

Remark 5.2.23. Suppose we say that an endofunctor $t : \mathfrak{X} \rightarrow \mathfrak{X}$ is *bianimable* if it preserves sifted colimits and the right Kan extension of $\delta t|_{\mathcal{A}} : \mathcal{A} \rightarrow \mathfrak{X}$ along $\mathcal{A} \hookrightarrow \mathfrak{A}$ preserves bar constructions. Let $\text{End}^{\text{banibl}}(\mathfrak{X}) \subseteq \text{End}(\mathfrak{X})$ and $\text{End}^{\text{banid}}(\mathfrak{X}) \subseteq \text{End}(\mathfrak{X})$ denote the full subcategories spanned by the bianimable endofunctors of $\mathfrak{X}$ and the bianimated endofunctors of $\mathfrak{X}$ (Definition 5.2.20), respectively. It follows from Proposition 4.1.15 that we have an equivalence of ∞ -categories

$$(5.2.24) \quad \text{End}^{\text{banibl}}(\mathfrak{X}) \xrightarrow{\sim} \text{End}^{\text{banid}}(\mathfrak{X}),$$

sending a bianimable endofunctor $t : \mathfrak{X} \rightarrow \mathfrak{X}$ to its *right-left extension* (Remark 5.2.9), i.e. the left Kan extension along the inclusion $\mathfrak{A} \hookrightarrow \mathfrak{X}$ of the right Kan extension along the inclusion $\mathcal{A} \hookrightarrow \mathfrak{A}$ of $t|_{\mathcal{A}}$; the inverse equivalence sends a bianimated endofunctor $\mathfrak{t} : \mathfrak{X} \rightarrow \mathfrak{X}$ to the left Kan extension of $\mathfrak{t}|_{\mathcal{A}}$ along the inclusion $\mathcal{A} \hookrightarrow \mathfrak{X}$.

Proposition 5.2.22 implies that the equivalence (5.2.24) respects monad structures, i.e. that there is an equivalence between bianimable monads on $\mathfrak{X}$ and bianimated monads on $\mathfrak{X}$ which is given on underlying endofunctors as above. As was explained to us by Brantner, this fact may be seen more directly: (5.2.24) extends to an equivalence of ∞ -operads.

To repeat from the introduction to this chapter, from the point of view of this remark, the primary thing our theoretical approach adds to the situation is an explicit presentation of the ∞ -category of

modules over the right-left extension $\mathfrak{t}$ of a bianimable monad t on $\mathcal{X}$, in terms of the category of a modules over t itself.

§5.3. Abelian categories I. In this section, we show that the bianimation construction of §5.2 recovers the usual derived ∞ -category for abelian inputs. We continue with the setup of Notation 5.2.1; in particular, $\mathcal{X}$ denotes a locally small, cocomplete, abelian category generated by compact 1-projective objects, and $\mathfrak{X}$ denotes $\mathrm{Spt}(\mathrm{Ani}(\mathcal{X})) \simeq \mathrm{D}(\mathcal{X})$.

Proposition 5.3.1. *$\mathcal{X}$ is bianimable over itself, and the structure map $u : \mathrm{bAni}(\mathcal{X}/\mathcal{X}) \rightarrow \mathfrak{X}$ is an equivalence.*

Proof. Theorem 5.2.10.1 identifies $\mathrm{bAni}(\mathcal{X}/\mathcal{X})$ with the $\mathfrak{A}$ -theorization of $\mathfrak{X}_{\geq 0}$ over $\mathfrak{X}$. That this is $\mathfrak{X}$ itself is a special case of Example 4.4.17. $\square$

Corollary 5.3.2. *Let $\mathcal{Y}$ be an abelian, algebraic category over $\mathcal{X}$. Then $\mathcal{Y}$ is bianimable over $\mathcal{X}$, and there is a unique equivalence $\mathrm{bAni}(\mathcal{Y}/\mathcal{X}) \xrightarrow{\sim} \mathrm{D}(\mathcal{Y})$ fitting into a commutative square*

$$\begin{array}{ccc} \mathrm{Ani}(\mathcal{Y}) & \xhookrightarrow{\theta} & \mathrm{bAni}(\mathcal{Y}/\mathcal{X}) \\ \sim \downarrow & & \downarrow \sim \\ \mathrm{D}(\mathcal{Y})_{\geq 0} & \hookrightarrow & \mathrm{D}(\mathcal{Y}), \end{array}$$

where θ is as in Theorem 5.2.10, the lower horizontal arrow is the inclusion, and the left vertical equivalence is as in Example 5.1.9.

Proof. Using the identification $\mathrm{bAni}(\mathcal{Y}/\mathcal{Y}) \simeq \mathrm{bAni}(\mathcal{Y}/\mathcal{X})$ of Example 5.2.17, we may assume without loss of generality that $\mathcal{Y} = \mathcal{X}$. The claimed bianimability and equivalence are then given by Proposition 5.3.1, the commutative square by Theorem 5.2.10, and the uniqueness from the fact that that each horizontal embedding defines the connective part of a right complete t -structure on its target. $\square$

Corollary 5.3.3. *Let $\{A_i\}_{i \in \mathbf{I}}$ be a detecting set of compact 1-projective objects in $\mathcal{X}$, and let $\rho : \mathcal{X} \rightarrow \mathrm{Ab}^{\mathbf{I}}$ and $\mathfrak{r} : \mathfrak{X} \rightarrow \mathrm{Spt}^{\mathbf{I}}$ be the functors corepresented by this set of objects. Then the commutative square*

$$\begin{array}{ccc} \mathcal{X} & \xhookrightarrow{\delta} & \mathfrak{X} \\ \rho \downarrow & & \downarrow \mathfrak{r} \\ \mathrm{Ab}^{\mathbf{I}} & \xhookrightarrow{\mathrm{H}} & \mathrm{Spt}^{\mathbf{I}} \end{array}$$

(where H denotes the Eilenberg–MacLane embedding) exhibits $\mathfrak{X} \simeq \mathrm{D}(\mathcal{X})$ as the $\mathrm{A}^{\mathbf{I}}$ -theorization, algebraization, and strong algebraization of $\mathcal{X}$ over $\mathrm{Spt}^{\mathbf{I}}$.

Proof. For the $\mathrm{A}^{\mathbf{I}}$ -theorization claim, combine Proposition 5.3.1 and Example 5.2.18. Since $\mathfrak{r}$ is strongly algebraic, the algebraization and strong algebraization claims follow (Remark 2.3.12). $\square$

§5.4. Commutative rings. For this section, we augment our setup from Notation 5.2.1 by assuming that our abelian, classically algebraic category $\mathcal{X}$ is equipped with a presentable symmetric monoidal structure. This induces a presentable symmetric monoidal structure on its animation $\mathfrak{X}_{\geq 0} = \mathrm{Ani}(\mathcal{X})$ and then also on the stabilization $\mathfrak{X} = \mathrm{Spt}(\mathfrak{X}_{\geq 0})$, such that, in the adjunctions

$$\pi_0 : \mathfrak{X}_{\geq 0} \rightleftarrows \mathcal{X} : \eta_{\mathrm{Ani}}, \quad \iota : \mathfrak{X}_{\geq 0} \rightleftarrows \mathfrak{X} : \tau_{\geq 0},$$

the left adjoints carry canonical symmetric monoidal structures, and hence the right adjoints lax symmetric monoidal structures.

We have associated ∞ -categories of commutative algebra objects, fitting into cartesian squares

$$\begin{array}{ccccc} \mathrm{CAlg}(\mathcal{X}) & \hookrightarrow & \mathrm{CAlg}(\mathfrak{X}_{\geq 0}) & \hookrightarrow & \mathrm{CAlg}(\mathfrak{X}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{X} & \xrightarrow{\eta_{\mathrm{Ani}}} & \mathfrak{X}_{\geq 0} & \xrightarrow{\iota} & \mathfrak{X}, \end{array}$$

where the vertical arrows are the forgetful functors, which are algebraic, and the left (resp. right) square is horizontally left (resp. right) adjointable. We also have a stricter notion of algebra in $\mathfrak{X}_{\geq 0}$, via animation:

Construction 5.4.1. As $\mathrm{CAlg}(\mathcal{X})$ is algebraic over $\mathcal{X}$, we may consider its animation $\mathrm{Ani}(\mathrm{CAlg}(\mathcal{X}))$, which carries an the animated forgetful functor to $\mathfrak{X}_{\geq 0}$ (as in Construction 5.1.11). Furthermore, we have a unique functor

$$\Theta_{\geq 0} : \mathrm{Ani}(\mathrm{CAlg}(\mathcal{X})) \rightarrow \mathrm{CAlg}(\mathfrak{X}_{\geq 0})$$

over $\mathfrak{X}_{\geq 0}$ whose precomposition with $\eta_{\mathrm{Ani}} : \mathrm{CAlg}(\mathcal{X}) \hookrightarrow \mathrm{Ani}(\mathrm{CAlg}(\mathcal{X}))$ identifies with the embedding $\mathrm{CAlg}(\mathcal{X}) \hookrightarrow \mathrm{CAlg}(\mathfrak{X}_{\geq 0})$ (this may be thought of in terms of the universal property of algebraization, in view of Proposition 5.1.14). The functor $\Theta_{\geq 0}$ preserves colimits, and so is strongly algebraic.

Mathew introduced a nonconnective enlargement of this animation; let us review the account from [35]:

Construction 5.4.2. [35, §4.2] constructs an ∞ -category $\mathrm{DAlg}(\mathfrak{X})$, equipped with a strongly algebraic functor $\Theta : \mathrm{DAlg}(\mathfrak{X}) \rightarrow \mathrm{CAlg}(\mathfrak{X})$ that fits into a cartesian square

$$\begin{array}{ccc} \mathrm{Ani}(\mathrm{CAlg}(\mathcal{X})) & \hookrightarrow & \mathrm{DAlg}(\mathfrak{X}) \\ \Theta_{\geq 0} \downarrow & & \downarrow \Theta \\ \mathrm{CAlg}(\mathfrak{X}_{\geq 0}) & \hookrightarrow & \mathrm{CAlg}(\mathfrak{X}). \end{array}$$

We outline that construction again here, in particular to clarify that the further assumptions made in op. cit. on the symmetric monoidal structure of $\mathcal{X}$ (articulated in the notion of a *derived algebraic context* [35, Definition 4.2.1]) are in fact unnecessary.³

Let $\mathrm{Sym}_{\mathfrak{X}} : \mathfrak{X} \rightarrow \mathfrak{X}$ and $\mathrm{Sym}_{\mathfrak{X}_{\geq 0}}, \mathrm{LSym}_{\mathfrak{X}_{\geq 0}} : \mathfrak{X}_{\geq 0} \rightarrow \mathfrak{X}_{\geq 0}$ denote the monads associated to the forgetful functors from $\mathrm{CAlg}(\mathfrak{X})$ to $\mathfrak{X}$ and from $\mathrm{CAlg}(\mathfrak{X}_{\geq 0})$ and $\mathrm{Ani}(\mathrm{CAlg}(\mathcal{X}))$ to $\mathfrak{X}_{\geq 0}$, respectively. The functor $\Theta_{\geq 0}$ corresponds to a map of monads $\theta_{\geq 0} : \mathrm{Sym}_{\mathfrak{X}_{\geq 0}} \rightarrow \mathrm{LSym}_{\mathfrak{X}_{\geq 0}}$. Constructing the ∞ -category $\mathrm{DAlg}(\mathfrak{X})$ and functor Θ amounts to constructing an extension of $\mathrm{LSym}_{\mathfrak{X}_{\geq 0}}$ to a monad $\mathrm{LSym}_{\mathfrak{X}} : \mathfrak{X} \rightarrow \mathfrak{X}$ along with an extension of $\theta_{\geq 0}$ to a map of monads $\theta : \mathrm{Sym}_{\mathfrak{X}} \rightarrow \mathrm{LSym}_{\mathfrak{X}}$.

Let $\mathfrak{E}_{\geq 0} \subseteq \mathrm{End}(\mathfrak{X}_{\geq 0})$ be the full subcategory spanned by endofunctors that preserve sifted colimits and let $\mathfrak{E}'_{\geq 0} \subseteq \mathfrak{E}_{\geq 0}$ be the further full subcategory spanned by endofunctors that additionally carry $\mathcal{A}$ into $\mathcal{X}$. Then restriction along the inclusion $\mathcal{A} \hookrightarrow \mathfrak{X}_{\geq 0} = \mathcal{P}_{\Sigma}(\mathcal{A})$ determines equivalences

$$\mathfrak{E}_{\geq 0} \simeq \mathrm{Fun}(\mathcal{A}, \mathfrak{X}_{\geq 0}), \quad \mathfrak{E}'_{\geq 0} \simeq \mathrm{Fun}(\mathcal{A}, \mathcal{X}),$$

under which the inclusion $\mathfrak{E}'_{\geq 0} \hookrightarrow \mathfrak{E}_{\geq 0}$ corresponds to composition with the inclusion $\eta_{\mathrm{Ani}} : \mathcal{X} \rightarrow \mathfrak{X}_{\geq 0}$. It follows that $\mathfrak{E}'_{\geq 0}$ is a reflective subcategory of $\mathfrak{E}_{\geq 0}$, the left adjoint to the inclusion corresponding to composition with $\pi_0 : \mathfrak{X}_{\geq 0} \rightarrow \mathcal{X}$. This reflection is compatible with the composition monoidal structure on $\mathfrak{E}_{\geq 0}$ (see [35, Lemma 4.2.17]) and hence preserves monad structures; concretely, the induced monoidal structure on $\mathfrak{E}_{\geq 0}$ agrees with the one obtained by further identifying $\mathrm{Fun}(\mathcal{A}, \mathcal{X})$ with the full subcategory of $\mathrm{End}(\mathcal{X})$ spanned by sifted colimit preserving functors. As an example, the map of monads $\theta_{\geq 0} : \mathrm{Sym}_{\mathfrak{X}_{\geq 0}} \rightarrow \mathrm{LSym}_{\mathfrak{X}_{\geq 0}}$ is a $\mathfrak{E}'_{\geq 0}$ -reflection.

The monad $\mathrm{Sym}_{\mathfrak{X}_{\geq 0}}$ refines to a filtered monad $\mathrm{Sym}_{\mathfrak{X}_{\geq 0}}^{\leq *}$ [35, Definition 4.1.2 and Construction 4.1.8]. Applying $\mathfrak{E}'_{\geq 0}$ -reflection, we obtain a map of filtered monads $\theta_{\geq 0}^{\leq *} : \mathrm{Sym}_{\mathfrak{X}_{\geq 0}}^{\leq *} \rightarrow \mathrm{LSym}_{\mathfrak{X}_{\geq 0}}^{\leq *}$ refining $\theta_{\geq 0}$. Each

³This point was communicated to me by Yutaro Mikami a few years ago, but I failed to process it at the time and then forgot about it while revising [35] for publication. For all of this negligence, I apologize.

piece of these filtrations is polynomial, in the excisive sense of Goodwillie, as a functor from $\mathfrak{X}_{\geq 0}$ to $\mathfrak{X}$. Let $\mathfrak{E}''_{\geq 0} \subseteq \mathfrak{E}_{\geq 0}$ be the full subcategory spanned by those sifted colimit preserving endofunctors of $\mathfrak{X}_{\geq 0}$ whose composition with the inclusion $\mathfrak{X}_{\geq 0} \hookrightarrow \mathfrak{X}$ is excisively polynomial, and let $\mathfrak{E}'' \subseteq \text{End}(\mathfrak{X})$ be the full subcategory of endofunctors of $\mathfrak{X}$ that preserve sifted colimits, are excisively polynomial, and preserve the full subcategory $\mathfrak{X}_{\geq 0}$. Then the restriction functor $\mathfrak{E}'' \rightarrow \mathfrak{E}''_{\geq 0}$ is an equivalence [35, Proposition 4.2.15]. Noting that $\mathfrak{E}''$ and $\mathfrak{E}''_{\geq 0}$ are closed under composition, we deduce that $\theta_{\geq 0}^{\leq *}$ extends uniquely to a map of filtered monads $\theta^{\leq *} : \text{Sym}_{\mathfrak{X}}^{\leq *} \rightarrow \text{LSym}_{\mathfrak{X}}^{\leq *}$ on $\mathfrak{X}$ for which the filtered pieces of the target are sifted colimit preserving, excisively polynomial functors (as they are for the source). Passing to underlying monads completes the construction.

The main result of this section is that Mathew's construction may be recharacterized as a bianimation in the sense of §5.2.

Theorem 5.4.3. *$\text{CAlg}(\mathcal{X})$ is bianimable over $\mathcal{X}$, and the embedding $\text{Ani}(\text{CAlg}(\mathcal{X})) \hookrightarrow \text{DAlg}(\mathfrak{X})$ of Construction 5.4.2 extends uniquely to an equivalence of ∞ -categories*

$$\text{bAni}(\text{CAlg}(\mathcal{X})/\mathcal{X}) \xrightarrow{\sim} \text{DAlg}(\mathfrak{X}).$$

We make some commentary on the meaning of the statement before proving it.

Remark 5.4.4. Let R be a commutative ring. Let's unravel the construction of $\text{bAni}(\text{CAlg}(\mathcal{X})/\mathcal{X})$ when $\mathcal{X} = \text{Mod}_R^{\vee}$, so that $\mathfrak{X} \simeq \text{Mod}_R$ (Example 5.2.3), these being equipped with their standard symmetric monoidal structures. Let $\mathcal{A}' \subseteq \text{Mod}_R^{\vee}$ be the full subcategory of finite free R -modules; note that, by Remark 5.2.16, in the construction we may use $\mathcal{A}'$ in place of $\mathcal{A}$ (which is the subcategory of finite projective R -modules) and correspondingly the full subcategory $\mathfrak{A}' = \mathbf{A}_R \subseteq \text{Mod}_R$ generated under desuspensions by $\mathcal{A}'$ in place of $\mathfrak{A}$.

Set $\text{CAlg}_R^{\vee} := \text{CAlg}(\text{Mod}_R^{\vee})$, and let us write $\mathbb{G}_{\mathbf{a}|R} : \text{CAlg}_R^{\vee} \rightarrow \text{Mod}_R^{\vee}$ for the forgetful (i.e. underlying additive group) functor. The $\text{Mod}_R|\mathbf{A}_R$ -theory

$$\mathfrak{T} := \text{Str}_{\mathbf{A}_R}(\text{CAlg}_R^{\vee}) \subseteq \text{Fun}(\text{CAlg}_R^{\vee}, \text{Spc})$$

is generated under finite products by the functors $B^n \mathbb{G}_{\mathbf{a}|R} = K(\mathbb{G}_{\mathbf{a}|R}, n)$ for $n \in \mathbb{Z}_{\geq 0}$, i.e. the functors carrying a commutative R -algebra A to the Eilenberg–MacLane space $K(A, n)$ associated to the underlying additive group of A . The bianimation/theorization

$$\text{bAni}(\text{CAlg}_R^{\vee}/\text{Mod}_R^{\vee}) \simeq (\text{CAlg}_R^{\vee})^{\mathbf{A}_R\text{-th}/\text{Mod}_R} \subseteq \text{Fun}(\mathfrak{T}, \text{Spc})$$

consists of those functors from $\mathfrak{T}$ to Spc that preserve finite products and the looping equivalences $K(\mathbb{G}_{\mathbf{a}|R}, n) \simeq \Omega K(\mathbb{G}_{\mathbf{a}|R}, n+1)$; equivalently, it is characterized by the following universal property as a cocomplete ∞ -category: for any other cocomplete ∞ -category $\mathcal{E}$, colimit preserving functors from $\text{bAni}(\text{CAlg}_R^{\vee}/\text{Mod}_R^{\vee})$ to $\mathcal{E}$ are equivalent to functors from $\mathfrak{T}^{\text{op}}$ to $\mathcal{E}$ that carry finite products in $\mathfrak{T}$ to finite coproducts and the aforementioned loopings in $\mathfrak{T}$ to suspensions (which may be regarded as versions of the Künneth and Eilenberg–Moore formulas).

As our notation here indicates, it is natural to think of this construction in algebro-geometric terms: we may regard $\mathfrak{T}$ as an ∞ -category of stacks, in the sense reviewed in Ch. A. More precisely, let us make the canonical identification $\text{CAlg}_R^{\vee} \simeq \text{CRing}_R$; this induces an identification $\text{Fun}(\text{CAlg}_R^{\vee}, \text{Spc}) \simeq \text{Fun}(\text{CRing}, \text{Spc})_{/\text{Spec}(R)}$, under which $\mathfrak{T}$ corresponds to the full subcategory of $\text{PStk}_{/\text{Spec}(R)}$ generated under finite products by $B^n \mathbb{G}_{\mathbf{a}} \times \text{Spec}(R)$ for $n \in \mathbb{Z}_{\geq 0}$ (as in Notation A.1.1, $\text{PStk} = \mathcal{P}^{\vee}(\text{CRing})$ is the ∞ -category of prestacks and Spec denotes the Yoneda embedding; and here $B^n \mathbb{G}_{\mathbf{a}}$ refers to $B^n \mathbb{G}_{\mathbf{a}|\mathbb{Z}}$). From this perspective, the functor $\Gamma_{\mathbb{G}_{\mathbf{a}|R}} : \mathfrak{T} \rightarrow \text{Mod}_R$ (which is also the restriction of the structure map $u : \text{bAni}(\text{CAlg}_R^{\vee}/\text{Mod}_R^{\vee}) \rightarrow \text{Mod}_R$ along the Yoneda embedding $\mathfrak{T} \hookrightarrow \text{bAni}(\text{CAlg}_R^{\vee}/\text{Mod}_R^{\vee})$, by Proposition 2.2.27) may be interpreted as the functor of derived global sections. That is, we have a commutative

diagram

$$\begin{array}{ccc}
\mathfrak{T}^{\text{op}} & \xrightarrow{\quad} & \text{PStk}_{/\text{Spec}(R)}^{\text{op}} \\
& \searrow \Gamma_{\mathbb{G}_a|R} & \swarrow \Gamma(-; \mathcal{O}_{(-)}) \\
& \text{Mod}_R &
\end{array}$$

(the right diagonal arrow being as in Example A.1.4, sending a prestack X to the global sections of its structure sheaf, $\Gamma(X; \mathcal{O}_X) \simeq \lim_{A \in \text{CRing}, x \in X(A)} A$).

In summary, $\text{bAni}(\text{CAlg}_R^{\heartsuit}/\text{Mod}_R^{\heartsuit})$ is a monadic ∞ -category over Mod_R for which the full subcategory of free objects on $A_R \subseteq \text{Mod}_R$ identifies contravariantly with the above ∞ -category of stacks $\mathfrak{T}$, and the underlying R -modules of these free objects are given by the derived global sections of the corresponding stacks.

Remark 5.4.5. Taking $R = \mathbb{Z}$ in Remark 5.4.4, we see that Theorem 5.4.3 specializes to Theorem 0.1.1 from the introduction. Applying Theorem 5.2.10.2, it also gives the variant of Theorem 0.1.4 with $\text{Mod}_{\mathbb{Z}}$ in place of Spt ; and invoking Example 5.2.18, it in fact gives Theorem 0.1.4 itself.

Remark 5.4.6. By definition of bianimability, Theorem 5.4.3 tells us that sifted colimits in the ∞ -category $\text{bAni}(\text{CAlg}(\mathfrak{X})/\mathfrak{X}) = \text{CAlg}(\mathfrak{X})^{\mathfrak{A}\text{-th}/\mathfrak{X}}$ are preserved by the structure map to $\mathfrak{X}$. The rest of the result implies more about colimits in $\text{bAni}(\text{CAlg}(\mathfrak{X})/\mathfrak{X})$: namely, the functor $v' : \text{bAni}(\text{CAlg}(\mathfrak{X})/\mathfrak{X}) \rightarrow \text{CAlg}(\mathfrak{X})$ induced (via the universal property of $\mathfrak{A}$ -theorization) by the embedding $v : \text{CAlg}(\mathfrak{X}) \hookrightarrow \text{CAlg}(\mathfrak{X})$ is in fact strongly algebraic, i.e. preserves all colimits, because it identifies with the functor $\Theta : \text{DAlg}(\mathfrak{X}) \rightarrow \text{CAlg}(\mathfrak{X})$ of Construction 5.4.2.

Let's discuss an alternative proof of this in the case that $\mathfrak{X} = \text{Mod}_R^{\heartsuit}$ for R a commutative ring, in particular giving an alternative proof of bianimability in this case. Setting $\text{CAlg}_R := \text{CAlg}(\text{Mod}_R)$, it suffices by Theorem 4.2.9.2 to show that the functor $\Gamma_v : \mathfrak{T}^{\text{op}} \rightarrow \text{CAlg}_R$ satisfies the A_R -theoretical Eilenberg–Moore formula (and since pushouts in CAlg_R are given by relative tensor products, the Eilenberg–Moore formula in this context is of the conventional form). Let us approach this from the algebro-geometric perspective explained in Remark 5.4.4, identifying $\mathfrak{T}$ with a full subcategory of $\text{PStk}_{/\text{Spec}(R)}$ and Γ_v with the global sections functor $\Gamma(-; \mathcal{O}_{(-)})$ of Example A.1.4. We would like to show that:

1. for $X, Y \in \mathfrak{T}$, the canonical map

$$\Gamma(X; \mathcal{O}_X) \otimes_R \Gamma(Y; \mathcal{O}_Y) \rightarrow \Gamma(X \times_{\text{Spec}(R)} Y; \mathcal{O}_{X \times_{\text{Spec}(R)} Y})$$

in CAlg_R is an equivalence;

2. for each nonnegative integer n , the canonical map

$$R \otimes_{\Gamma(B^{n+1}\mathbb{G}_{a|R}; \mathcal{O}_{B^{n+1}\mathbb{G}_{a|R}})} R \rightarrow \Gamma(B^n\mathbb{G}_{a|R}; \mathcal{O}_{B^n\mathbb{G}_{a|R}})$$

in CAlg_R is an equivalence.

Here we have broken up the A_R -theoretical Eilenberg–Moore formula into the A_R -theoretical Künneth formula and the Eilenberg–Moore formula for the generating objects $B^n\mathbb{G}_{a|R}$ (Remark 4.2.7).

This problem may be solved by placing it in the broader context of quasicoherent sheaves on geometric stacks; we will refer to our review of the necessary theory in Ch. A. We claim first that each $X \in \mathfrak{T}$ is a geometric stack; in fact, we claim that the projection map $X \rightarrow \text{Spec}(R)$ is geometrically flat (Definition A.2.2). That they are (fpqc) stacks follows from the fact that they are corepresentable by $\mathbb{E}_{\infty}\text{-}\mathbb{Z}$ -algebras—namely, the free $\mathbb{E}_{\infty}$ -algebras on the objects of $A_{\mathbb{Z}} \subseteq \text{Mod}_{\mathbb{Z}}$ —together with the fact that the identity functor $\text{id} : \text{CAlg}_{\mathbb{Z}} \rightarrow \text{CAlg}_{\mathbb{Z}}$ satisfies fpqc descent [29, Theorem D.6.3.5]. Concerning the condition that X is geometrically flat over $\text{Spec}(R)$, this is stable under finite products (Corollary A.2.5), so it suffices to check that it holds in the case $X = B^n\mathbb{G}_{a|R}$ for any nonnegative integer n . This can be proved by induction on n ; indeed, this is a case of Example A.2.7, for the flat affine commutative group scheme $\mathbb{G}_{a|R}$ over R , corresponding to the bicommutative Hopf algebra $A := R[t]$ with coproduct $\Delta : A \rightarrow A \otimes_R A$ given by the formula $\Delta(t) = t \otimes 1 + 1 \otimes t$.

We will now appeal to Corollary A.3.7 to prove 1 and 2. To begin, we observe that it suffices to treat the case $R = \mathbb{Z}$: indeed, in this case, for any commutative ring R' and $X \in \mathfrak{T}$, we may apply Corollary A.3.7

to the cartesian diagram

$$\begin{array}{ccc} X \times \mathrm{Spec}(R') & \longrightarrow & \mathrm{Spec}(R') \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathrm{Spec}(\mathbb{Z}) \end{array}$$

(R' being a unipotent $\mathbb{Z}$ -module by Example A.3.3.3) to get that the canonical map $R' \otimes_{\mathbb{Z}} \Gamma(X; \mathcal{O}_X) \rightarrow \Gamma(X \times \mathrm{Spec}(R'); \mathcal{O}_{X \times \mathrm{Spec}(R')})$ is an equivalence.

So assume that $R = \mathbb{Z}$. For the Künneth formula 1, we apply Corollary A.3.7 to the cartesian diagram

$$\begin{array}{ccc} X \times Y & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathrm{Spec}(\mathbb{Z}) \end{array}$$

(again using from Example A.3.3.3 that every left bounded $\mathbb{Z}$ -module is unipotent). For the Eilenberg–Moore formula 2, we are considering the cartesian diagram

$$\begin{array}{ccc} B^n \mathbb{G}_a & \xrightarrow{p} & \mathrm{Spec}(\mathbb{Z}) \\ p \downarrow & & \downarrow e \\ \mathrm{Spec}(\mathbb{Z}) & \xrightarrow{e} & B^{n+1} \mathbb{G}_a, \end{array}$$

so we just need to verify that $e_*(\mathbb{Z}) \in \mathrm{QCoh}(B^{n+1} \mathbb{G}_a)$ is unipotent and then we may apply Corollary A.3.7 once more to finish. For $n \geq 1$, this follows from Example A.3.5: that is, $B^{n+1} \mathbb{G}_a$ is simply connected, and hence any left bounded quasicoherent sheaf on $B^{n+1} \mathbb{G}_a$ is unipotent.

For $n = 0$, we can argue concretely. Note first that $e : \mathrm{Spec}(\mathbb{Z}) \rightarrow B \mathbb{G}_a$ is 0-geometrically faithfully flat, and so, by Proposition A.2.10.3, we have

$$e^* e_*(\mathbb{Z}) \simeq p_* p^*(\mathbb{Z}) \simeq A \in \mathrm{Mod}_{\mathbb{Z}}^{\vee},$$

where $A = \mathbb{Z}[t]$ is the Hopf algebra from earlier. The functor e^* is t-exact (Proposition A.2.9.3) and conservative, so we deduce that $e_*(\mathbb{Z})$ lies in the heart $\mathrm{QCoh}(B \mathbb{G}_a)^{\vee}$. By Proposition A.2.9.2, we have an identification

$$\mathrm{QCoh}(B \mathbb{G}_a)^{\vee} \simeq \mathrm{QCoh}^{\vee}(B \mathbb{G}_a) \simeq \lim_{[m] \in \Delta} \mathrm{QCoh}^{\vee}(\mathbb{G}_a^{\times m}) \simeq \lim_{[m] \in \Delta} \mathrm{Mod}_{A^{\otimes_{\mathbb{Z}} m}}^{\vee}.$$

A straightforward analysis identifies this with the category of (classical) comodules over the Hopf algebra A , with $e_*(\mathbb{Z})$ corresponding to A regarded as a comodule over itself via Δ , and $\mathcal{O}_{B \mathbb{G}_a}$ corresponding to the trivial comodule $T \simeq \mathbb{Z}$. Thus, to exhibit the unipotence of $e_*(\mathbb{Z})$, it suffices by Remark A.3.2 to find a filtration of A -comodules

$$0 = M_{-1} \hookrightarrow M_0 \hookrightarrow M_1 \hookrightarrow M_2 \hookrightarrow \dots$$

such that $\mathrm{colim}(M_*) \simeq A$ and $M_k/M_{k-1} \simeq T$ for each $k \geq 0$. We may take M_k to be the subset of A consisting of polynomials of degree at most k . This completes the proof.

Proof of Theorem 5.4.3. By Proposition 5.2.22, it suffices to show that $\mathrm{DAlg}(\mathfrak{X})$ is a bianimated ∞ -category over $\mathfrak{X}$ in the sense of Definition 5.2.20, i.e. that the monad $\mathrm{LSym}_{\mathfrak{X}} : \mathfrak{X} \rightarrow \mathfrak{X}$ from Construction 5.4.2 carries $\mathcal{A}$ into $\mathfrak{X}$ and on $\mathfrak{A}$ preserves cobar constructions. The former is true by construction. For the latter, recall that $\mathrm{LSym}_{\mathfrak{X}}$ is the colimit of a sequence of excisively polynomial, sifted colimit preserving functors

$$\mathrm{LSym}_{\mathfrak{X}}^{\leq 0} \rightarrow \mathrm{LSym}_{\mathfrak{X}}^{\leq 1} \rightarrow \mathrm{LSym}_{\mathfrak{X}}^{\leq 2} \rightarrow \dots$$

By [12, Proposition 3.38], each functor in this sequence preserves cobar constructions on $\mathfrak{A}$ (which we recall is equivalent to its restriction to $\mathfrak{A}$ being right Kan extended from $\mathcal{A}$). Noting that this implies that $\mathrm{LSym}_{\mathfrak{X}}^{\leq k}(A)$ is a coconnective object of $\mathfrak{X}$ for any $k \geq 0$ and $A \in \mathfrak{A}$ (as we know this to be true for $A \in \mathcal{A}$ and coconnectivity is closed under limits), and recalling that totalizations commute with filtered colimits in the coconnective subcategory $\mathfrak{X}_{\leq 0}$, we conclude that $\mathrm{LSym}_{\mathfrak{X}}$ also preserves cobar constructions. $\square$

§5.5. Abelian categories II. Any Grothendieck abelian category $\mathcal{X}$ has a derived ∞ -category $D(\mathcal{X})$. Biamimation, as we have formulated it here, only recovers this construction for classically algebraic (i.e. compact 1-projectively generated) abelian categories, as discussed in §5.3. In this section, we discuss a different way to recover $D(\mathcal{X})$ via theorization, which applies more, though still not completely, generally. We recall the following class of Grothendieck abelian categories.

Definition 5.5.1. We say that an abelian category $\mathcal{X}$ is *coherent* if it is compactly generated and the full subcategory of $\mathcal{X}$ spanned by the compact objects is abelian, or equivalently, if there exists an equivalence of categories $\mathcal{X} \simeq \text{Ind}(\mathcal{A})$ for a small abelian category $\mathcal{A}$. (In some other sources, $\mathcal{X}$ is instead called locally coherent when this condition is satisfied.)

Example 5.5.2. A ring R is said to be *left coherent* if the abelian category $\text{Mod}_R^\vee$ of classical left R -modules is coherent in the sense of Definition 5.5.1. This is equivalent to the condition that every finitely generated left ideal of R is finitely presented.

Example 5.5.3. Here we will invoke some of the algebro-geometric notions reviewed in Ch. A. Let $X : \text{CRing} \rightarrow \text{Spc}$ be a stack and suppose given a faithfully flat map $x : \text{Spec}(R) \rightarrow X$ for R a coherent commutative ring; here faithful flatness is meant in the representable sense, i.e. what is called 0-geometric faithful flatness in Definition A.2.1. That is, X is a 1-geometric stack, which in addition may be called *locally coherent*. Let us consider the case that $\mathcal{X} = \text{QCoh}(X)^\vee$.

Note that $x^* : \text{QCoh}(X) \rightarrow \text{Mod}_R$ is t-exact (Proposition A.2.9.3), hence restricts to an exact functor $x^* : \mathcal{X} = \text{QCoh}(X)^\vee \rightarrow \text{Mod}_R^\vee$. Let $\mathcal{A} \subseteq \mathcal{X}$ be the full subcategory spanned by those discrete quasicoherent sheaves $\mathcal{F}$ on X such that $x^*(\mathcal{F})$ is a finitely presented R -module. It follows from Example 5.5.2 that $\mathcal{A}$ is an abelian subcategory of $\mathcal{X}$. Moreover, every object of $\mathcal{A}$ is compact in $\mathcal{X}$. Thus, if $\mathcal{A}$ generates $\mathcal{X}$ under colimits, then $\mathcal{X}$ is coherent; this condition holds in particular when the stack X satisfies the Adams condition (see [29, §9.3.4]).

There is a variant of the derived ∞ -category that we'll meet more naturally in this setting.

Notation 5.5.4. Let $\mathcal{X}$ be a Grothendieck abelian category. We let $\mathcal{X}^{\text{inj}}$ denote the full subcategory of $\mathcal{X}$ spanned by the injective objects, and we let $\check{D}(\mathcal{X})$ be the *unseparated derived ∞ -category* of $\mathcal{X}$, as in [29, Definition C.5.8.2], i.e. the differential graded nerve of the category of chain complexes in $\mathcal{X}^{\text{inj}}$.

Remark 5.5.5. We recall a few points from [29, §C.5] concerning the close relationship among $\check{D}(\mathcal{X})$, $D(\mathcal{X})$, and the left completion $\hat{D}(\mathcal{X})$ for a Grothendieck abelian category $\mathcal{X}$ (see [26, §1.2.1] for the notion of left completion):

1. $\check{D}(\mathcal{X})$ also carries a t-structure [29, Notation C.5.8.4], and immediate from the definitions is a t-exact, fully faithful inclusion $D(\mathcal{X}) \hookrightarrow \check{D}(\mathcal{X})$ that induces an equivalence $D(\mathcal{X})_{\leq 0} \simeq \check{D}(\mathcal{X})_{\leq 0}$, and hence induces an equivalence on left completions and on hearts.
2. The t-structures on $\check{D}(\mathcal{X})$, $D(\mathcal{X})$, and $\hat{D}(\mathcal{X})$ are right complete and compatible with filtered colimits, so that they are determined by their connective subcategories, which are Grothendieck prestable ∞ -categories.
3. $\check{D}(\mathcal{X})_{\geq 0}$ is anticomplete, in the sense of [29, Definition C.5.5.4], and may be recovered as the anticompletion of $D(\mathcal{X})_{\geq 0}$ or $\hat{D}(\mathcal{X})_{\geq 0}$ (see [29, Corollary C.5.5.11, Theorem C.5.8.8, Proposition C.5.9.2]).
4. $D(\mathcal{X})_{\geq 0}$ is separated, in the sense of [29, Definition C.1.2.12], and the embedding $D(\mathcal{X})_{\geq 0} \hookrightarrow \check{D}(\mathcal{X})_{\geq 0}$ of 1 identifies $D(\mathcal{X})_{\geq 0}$ with the separated quotient of $\check{D}(\mathcal{X})_{\geq 0}$ (see [29, Proposition C.3.6.1, Theorem C.5.4.9, Corollary C.5.8.9]).

We may now state the main result of the section.

Theorem 5.5.6. *Let $\mathcal{X}$ be a coherent abelian category. Let $\{A_i\}_{i \in I}$ be a set of compact objects of $\mathcal{X}$ satisfying the following property:*

- (*) *In $\mathcal{X}$, any compact object admits a finite resolution by objects that can be written as filtered colimits of objects in $\{A_i\}_{i \in I}$.*

Let $\rho : \mathcal{X} \rightarrow \mathbf{Ab}^I$ and $\tau : \check{D}(\mathcal{X}) \rightarrow \mathbf{Spt}^I$ be the functors corepresented by this set of objects. Then the commutative diagram

$$\begin{array}{ccc} \mathcal{X}^{\text{inj}} & \xhookrightarrow{e} & \check{D}(\mathcal{X}) \\ \rho \downarrow & & \downarrow \tau \\ \mathbf{Ab}^I & \xhookrightarrow{H} & \mathbf{Spt}^I \end{array}$$

(where e is the embedding of the injective objects into the heart, and H is the Eilenberg–MacLane embedding) exhibits $\check{D}(\mathcal{X})$ as the A^I -theorization, algebraization, and strong algebraization of $\mathcal{X}^{\text{inj}}$ over $\mathbf{Spt}^I$.

Remark 5.5.7. In Theorem 5.5.6, we may take $\{A_i\}_{i \in I}$ to be a set representing every equivalence class of compact objects in $\mathcal{X}$, and then $(*)$ is tautologically satisfied.

Let us break up our proof of Theorem 5.5.6.

Lemma 5.5.8. *In the situation of Theorem 5.5.6, $\{A_i\}_{i \in I}$ is a set of stable compact generators of $\check{D}(\mathcal{X})$: that is, these objects are also compact in $\check{D}(\mathcal{X})$, and they generate $\check{D}(\mathcal{X})$ under colimits and desuspensions.*

Proof. By [29, Corollary C.6.5.9], $\check{D}(\mathcal{X})_{\geq 0}$ is a coherent Grothendieck prestable ∞ -category in the sense of [29, Definition C.6.5.1]. It is also anticomplete, so by [29, Proof of Theorem C.6.7.1], its truncated, almost compact objects are compact generators. By [29, Proof of Proposition C.6.5.7], an object of $\check{D}(\mathcal{X})_{\geq 0}$ is almost compact if and only if each of its homotopy groups is a compact object of $\mathcal{X}$. Combining the previous two sentences, we find that the compact objects of $\mathcal{X}$ also compactly generate $\check{D}(\mathcal{X})_{\geq 0}$, hence stably compactly generate $\check{D}(\mathcal{X})$ (since the t-structure on $\check{D}(\mathcal{X})$ is right complete and compatible with filtered colimits). We may now conclude by applying our hypothesis $(*)$. $\square$

Lemma 5.5.9. *Let $\mathcal{X}$ be a Grothendieck abelian category. Let $e : \mathcal{X}^{\text{inj}} \hookrightarrow \check{D}(\mathcal{X})_{\leq 0} = D(\mathcal{X})_{\leq 0}$ be the embedding into the heart. Then e is codense (Definition 1.2.5).*

Proof. Let $X \in \check{D}(\mathcal{X})_{\leq 0}$. We may represent X by a nonpositively graded chain complex $X_0 \rightarrow X_{-1} \rightarrow \dots$ in $\mathcal{X}^{\text{inj}}$. Equivalently, by the Dold–Kan correspondence, X is the limit in $\check{D}(\mathcal{X})$ of a cosimplicial object $Y^\bullet : \Delta \rightarrow \mathcal{X}^{\text{inj}}$. By Proposition 1.2.11, it suffices to show that this limit is preserved by the restricted Yoneda functor $j_e^\vee : \check{D}(\mathcal{X})_{\leq 0} \rightarrow \text{Fun}(\mathcal{X}^{\text{inj}}, \mathbf{Spc})^{\text{op}}$. In fact, this functor preserves all totalizations [29, Proposition C.5.7.1]. $\square$

Proof of Theorem 5.5.6. By Lemma 5.5.8, the functor $\tau : \check{D}(\mathcal{X}) \rightarrow \mathbf{Spt}^I$ is strongly algebraic. It therefore suffices to prove the A^I -theorization claim (Remark 2.3.12). Let $\mathfrak{l} : \mathbf{Spt}^I \rightarrow \check{D}(\mathcal{X})$ be the left adjoint of τ , which carries $\mathbb{S}(-i)$ to A_i for each $i \in I$. By Corollary 2.2.30, it's enough to show that the objects in the essential image of A^I under $\mathfrak{l}$ are e -corealized. This follows from Lemma 5.5.9, as these objects are coconnective. $\square$

Ch. 6. Even synthesis

Let R be an $\mathbb{E}_1$ -ring, and let Mod_R be the ∞ -category of (left) R -modules. In [33], Pstrągowski defines an ∞ -category $\text{Perf}_R^{\text{ev}} \subseteq \text{Mod}_R$ of *perfect even* R -modules, and then considers a certain ∞ -category of additive sheaves of spectra thereon, $\text{Shv}_\Sigma(\text{Perf}_R^{\text{ev}}; \text{Spt})$. Let us denote the latter by Syn_R^{ev} and refer to it as the ∞ -category of *synthetic R -modules*. In the case $R = \mathbb{S}$, [33, Proposition 3.17] shows that this recovers the ∞ -category of even MU-based synthetic spectra introduced by Pstrągowski in [34].

We recall two basic pieces of structure carried by Syn_R^{ev} : there is a “forgetful” functor $u : \text{Syn}_R^{\text{ev}} \rightarrow \text{Fil}$, where Fil denotes the ∞ -category of filtered spectra; and there is a “synthetic analogue” functor $\nu : \text{Mod}_R \rightarrow \text{Syn}_R^{\text{ev}}$. In fact, Syn_R^{ev} is introduced in [33] in order to study the composition $u \circ \nu$, which we will denote by $\text{fil}_{\text{pev}}^* : \text{Mod}_R \rightarrow \text{Fil}$ and refer to as the *perfect even filtration*, following the title and first paragraphs of [33] (whereas, in the remainder of [33], this construction is referred to simply as the even filtration and denoted fil_{ev}^* , motivated by the comparison results proven there to the even filtration for $\mathbb{E}_\infty$ -rings introduced by Hahn–R.–Wilson [17]).

The purpose of this chapter is to explain an alternative characterization of Syn_R^{ev} , as a theorization over Fil in the sense of Ch. 2. If R is connective, for example, it is a theorization (and algebraization) of the Postnikov filtration functor $\tau_{\geq 2*} : \text{Mod}_R^{\text{ev}} \rightarrow \text{Fil}$, where $\text{Mod}_R^{\text{ev}} \subseteq \text{Mod}_R$ is the ∞ -category of *even* R -modules, i.e. those with homotopy concentrated in even degrees; in other words, Syn_R^{ev} is an articulation of the structure carried by the Postnikov filtration of an even R -module. For general R , a small modification is necessary.

At the heart of this characterization is a method Pstrągowski explains in [33] for calculating the perfect even filtration of an R -module X via a certain kind of resolution of X by even R -modules. We’ll take the opportunity to reinterpret this in terms of yet another version of the even filtration construction, which is more directly analogous to that of [17] and hence will be denoted here by $\text{fil}_{\text{ev}}^* : \text{Mod}_R \rightarrow \text{Fil}$. What Pstrągowski’s method shows is that, up to completion, $\text{fil}_{\text{pev}}^*(X)$ identifies with $\text{fil}_{\text{ev}}^*(X)$ for a certain class of R -modules X . This isn’t true for all R -modules, however, and in cases where the two variants differ, it may also be of interest to consider $\text{fil}_{\text{ev}}^*(X)$.

Both Pstrągowski’s original approach to the synthetic ∞ -category Syn_R^{ev} and our approach via theorization go through in a more general context than that of the ∞ -category Mod_R . For instance, a variant of Pstrągowski’s construction in the setting of equivariant stable homotopy theory with respect to a finite abelian group is studied in work of Allen–Piessevaux [1]. After reviewing some relevant background on filtered spectra in §6.1, we’ll review the basic elements of Pstrągowski’s construction in an abstract setting which covers both of the aforementioned cases in §6.2. In §6.3, we’ll flesh out what was described in the previous paragraph, and in §6.4, we’ll deduce our alternative characterization of the synthetic ∞ -category.

§6.1. Filtered spectra.

Definition 6.1.1. A (*decreasingly*) *filtered spectrum* is a diagram of spectra

$$V = \{\dots \rightarrow V^1 \rightarrow V^0 \rightarrow V^{-1} \rightarrow \dots\};$$

we may also use the notation V^* to indicate such an object. We write Fil for the ∞ -category $\text{Fun}(\mathbb{Z}^{\text{op}}, \text{Spt})$ of filtered spectra. For each $n \in \mathbb{Z}$, we write $\mathbb{S}\{-n\}$ for the filtered spectrum corepresenting the evaluation functor $(-)^n : \text{Fil} \rightarrow \text{Spt}$.

Remark 6.1.2. In the subsequent sections of the chapter, we will generally be working with I -indexed collections of filtered spectra for a certain set I , i.e. we’ll be working with the ∞ -category Fil^I . For $i \in I$ and $n \in \mathbb{Z}$, we have an object $\mathbb{S}(-i)\{-n\} \in \text{Fil}^I$ corepresenting the functor $\{V_i^*\}_{i \in I} \mapsto V_i^n$ (recall from Convention 0.3.12 that we use the same minus sign convention, purely notationally, for the I -valued index). We’ll continue our discussion in this section just in the setting of Fil , with the understanding that it applies in a pointwise fashion to the more general I -indexed setting.

Notation 6.1.3. We write $\tau_{\geq 2*} : \text{Spt} \rightarrow \text{Fil}$ for the double speed Postnikov filtration functor. In the later

sections of the chapter, this functor will only be applied on the full subcategory of spectra with homotopy concentrated in even degrees.

Notation 6.1.4. We write $\delta : \text{Spt} \rightarrow \text{Fil}$ for the constant/diagonal functor. This admits both left and right adjoints, namely the colimit and limit functors $\text{colim} : \text{Fil} \rightarrow \text{Spt}$ and $\text{lim} : \text{Fil} \rightarrow \text{Spt}$. Both the counit map $\text{colim} \circ \delta \rightarrow \text{id}_{\text{Spt}}$ and unit map $\text{id}_{\text{Spt}} \rightarrow \text{lim} \circ \delta$ are equivalences, which is equivalent to δ being fully faithful. We say that a filtered spectrum is *constant* if it lies in the essential image of δ .

Definition 6.1.5. We say that a filtered spectrum V^* is *complete* if $\text{lim}(V^*) \simeq 0$, and we let $\widehat{\text{Fil}} \subseteq \text{Fil}$ denote the full subcategory spanned by the complete filtered spectra. This is a reflective subcategory, i.e. the inclusion $\widehat{\text{Fil}} \hookrightarrow \text{Fil}$ admits a left adjoint

$$\widehat{(-)} : \text{Fil} \rightarrow \widehat{\text{Fil}},$$

which we refer to as *completion*; it is characterized by a natural cofiber sequence

$$\delta(\text{lim}(V^*)) \rightarrow V^* \rightarrow \widehat{V^*}$$

associated to any filtered spectrum V^* , where the first map is the counit map for the adjunction between δ and lim and the second arrow is the unit map/reflection defining completion.

Example 6.1.6. For any spectrum E , the double speed Postnikov filtration $\tau_{\geq 2*}(E)$ is complete.

Notation 6.1.7. We set $\text{Gr} := \text{Spt}^{\mathbb{Z}}$ (here $\mathbb{Z}$ is regarded just as a set, or as a discrete ∞ -category), and we write

$$\text{gr}^* : \text{Fil} \rightarrow \text{Gr}$$

for the *associated graded* functor: for a filtered spectrum V , the associated graded spectrum $\text{gr}^*(V) = \{\text{gr}^n(V)\}_{n \in \mathbb{Z}}$ is defined by the formula $\text{gr}^n(V) := \text{cofib}(V^{n+1} \rightarrow V^n)$ for $n \in \mathbb{Z}$.

In the next sections, we will refer to certain filtered spectra using notation of the form $\text{fil}_?^*(-)$; in this case, we'll denote the associated graded object by $\text{gr}_?^*(-)$.

Remark 6.1.8. For a map of filtered spectra $\phi : V^* \rightarrow W^*$, the following conditions are equivalent:

1. the induced map on completions $\widehat{V^*} \rightarrow \widehat{W^*}$ is an equivalence;
2. the fiber of ϕ is constant, i.e. in the essential image of $\delta : \text{Spt} \rightarrow \text{Fil}$;
3. the induced map on associated graded objects $\text{gr}^*(V) \rightarrow \text{gr}^*(W)$ is an equivalence;
4. the induced square

$$\begin{array}{ccc} \delta(\text{lim}(V^*)) & \xrightarrow{\phi} & \delta(\text{lim}(W^*)) \\ \downarrow & & \downarrow \\ V^* & \xrightarrow{\phi} & W^* \end{array}$$

(where the vertical maps are the counit maps) is cartesian;

5. the induced square

$$\begin{array}{ccc} V^* & \xrightarrow{\phi} & W^* \\ \downarrow & & \downarrow \\ \delta(\text{colim}(V^*)) & \xrightarrow{\phi} & \delta(\text{colim}(W^*)) \end{array}$$

(where the vertical maps are the unit maps) is cartesian.

Furthermore, assuming that the above conditions are satisfied, then the following additional conditions are equivalent to each other:

6. ϕ is an equivalence;
7. the induced map $\text{lim}(V^*) \rightarrow \text{lim}(W^*)$ is an equivalence;
8. the induced map $\text{colim}(V^*) \rightarrow \text{colim}(W^*)$ is an equivalence.

§6.2. The perfect even filtration.

Notation 6.2.1. For the remainder of the chapter, we let $\mathcal{X}$ be a compactly generated stable ∞ -category, we fix a set $\mathbf{B} = \{B_i\}_{i \in I}$ of stable compact generators in $\mathcal{X}$, and we let $\mathcal{B} \subseteq \mathcal{X}$ be the full subcategory generated by $\mathbf{B}$ under even suspensions, even desuspensions, extensions, and retracts. We will try to be clear about when constructions depend on $\mathbf{B}$ or just on $\mathcal{B}$.

Example 6.2.2. Let R be an $\mathbb{E}_1$ -ring. Then we may take $\mathcal{X} = \text{Mod}_R$ and $\mathbf{B} = \{R\}$. In this case, $\mathcal{B}$ is the subcategory $\text{Perf}_R^{\text{ev}}$ of perfect even R -modules mentioned above, and the subsequent discussion will recover constructions of Pstrągowski from [33].

Example 6.2.3. Let G be a finite abelian group and suppose that $\mathcal{X} = \text{Spt}_G$, the ∞ -category of genuine G -spectra. Then we may take $\mathbf{B} = \{\Sigma_+^\infty G/H\}_{H \in \text{Sub}(G)}$, where $\text{Sub}(G)$ denotes the set of subgroups of G . There is another choice which appears to be more interesting, however: letting $\text{Rep}(G)$ be the Grothendieck group of finite dimensional complex representations of G , we may take the larger generating set $\mathbf{B}_{\text{Rep}(G)} = \{\Sigma_+^V \Sigma_+^\infty G/H\}_{H \in \text{Sub}(G), V \in \text{Rep}(G)}$. The full subcategory generated by $\mathbf{B}_{\text{Rep}(G)}$ under extensions and retracts (which then is already closed under even suspensions and even desuspensions) is the subcategory of *perfect pure* G -spectra considered by Allen–Piessevaux [1, Definition 8.1.1].

Remark 6.2.4. By [29, Proposition C.6.3.1], the generating set $\mathbf{B}$ determines a right complete t-structure $(\mathcal{X}_{\geq 0}^{\mathbf{B}}, \mathcal{X}_{\leq 0}^{\mathbf{B}})$, the connective part being generated under colimits and extensions by the objects of $\mathbf{B}$. We will say that $\mathbf{B}$ is *projective* if $\text{Ext}_{\mathcal{X}}^k(B_i, B_{i'}) \simeq 0$ for all $i, i' \in I$ and $k > 0$. Letting $\mathcal{B}_0 \subseteq \mathcal{X}_{\geq 0}^{\mathbf{B}}$ denote the full subcategory generated under finite sums by the objects of $\mathbf{B}$, this is equivalent to the restricted Yoneda functor $j_{\mathcal{X}_{\geq 0}^{\mathbf{B}}|\mathcal{B}_0} : \mathcal{X}_{\geq 0}^{\mathbf{B}} \rightarrow \mathcal{P}_{\Sigma}(\mathcal{B}_0)$ being an equivalence of ∞ -categories (Lemma 4.4.1 and Proposition 4.4.4). By right completeness of the t-structure, this then induces an equivalence

$$\mathcal{X} \simeq \text{Spt}(\mathcal{P}_{\Sigma}(\mathcal{B}_0)) \simeq \mathcal{P}_{\Sigma}(\mathcal{B}_0; \text{Spt}),$$

where $\mathcal{P}_{\Sigma}(\mathcal{B}_0; \text{Spt})$ denotes the full subcategory of $\text{Fun}(\mathcal{B}_0^{\text{op}}, \text{Spt})$ spanned by the finite product preserving functors.

Projectivity is satisfied in Example 6.2.2 if and only if R is connective. In Example 6.2.3, the generating set $\mathbf{B}$ is projective while $\mathbf{B}_{\text{Rep}(G)}$ is not.

The definitions of the synthetic ∞ -category and perfect even filtration from [33, §2] adapt immediately to the general context of Notation 6.2.1.

Construction 6.2.5. We define $\text{Syn}_{\mathcal{B}}(\mathcal{X})$ to be the full subcategory of $\text{Fun}(\mathcal{B}^{\text{op}}, \text{Spt})$ spanned by those functors $S : \mathcal{B}^{\text{op}} \rightarrow \text{Spt}$ that preserve finite products and those fiber/cofiber sequences that are preserved by the inclusion $\mathcal{B}^{\text{op}} \hookrightarrow \mathcal{X}^{\text{op}}$. This is a stable, presentable ∞ -category carrying a canonical t-structure $(\text{Syn}_{\mathcal{B}}(\mathcal{X})_{\geq 0}, \text{Syn}_{\mathcal{B}}(\mathcal{X})_{\leq 0})$, the coconnective part consisting of those objects $S \in \text{Syn}_{\mathcal{B}}(\mathcal{X})$ that are pointwise coconnective, i.e. such that the spectrum $S(B)$ is coconnective for each $B \in \mathcal{B}$.

As in [33, Notation 3.1 and Remark 3.2], we have a restricted Yoneda functor $\nu_{\mathcal{B}} : \mathcal{X} \rightarrow \text{Syn}_{\mathcal{B}}(\mathcal{X})$ and a restricted spectral Yoneda functor $Y_{\mathcal{B}} : \mathcal{X} \rightarrow \text{Syn}_{\mathcal{B}}(\mathcal{X})$. The two are related by the formula $\nu_{\mathcal{B}} \simeq \tau_{\geq 0} \circ Y_{\mathcal{B}}$, where $\tau_{\geq 0}$ denotes connective cover with respect to the t-structure of the previous paragraph; we write

$$c : \nu_{\mathcal{B}} \rightarrow Y_{\mathcal{B}}$$

for the connective cover map.

Remark 6.2.6. Beware that a connective object of $\text{Syn}_{\mathcal{B}}(\mathcal{X})$ is not necessarily pointwise connective, although this is true for the analogous t-structure on the larger full subcategory $\mathcal{P}_{\Sigma}(\mathcal{B}; \text{Spt}) \subseteq \text{Fun}(\mathcal{B}^{\text{op}}, \text{Spt})$ of finite product preserving functors (as well as on $\text{Fun}(\mathcal{B}^{\text{op}}, \text{Spt})$ itself). In other words, the inclusion functor $j : \text{Syn}_{\mathcal{B}}(\mathcal{X}) \hookrightarrow \mathcal{P}_{\Sigma}(\mathcal{B}; \text{Spt})$ is left t-exact but not right t-exact. Relatedly, the connective cover functor $\tau_{\geq 0} : \text{Syn}_{\mathcal{B}}(\mathcal{X}) \rightarrow \text{Syn}_{\mathcal{B}}(\mathcal{X})_{\geq 0}$ is not computed pointwise.

However, j admits a t-exact left adjoint $L : \mathcal{P}_{\Sigma}(\mathcal{B}; \text{Spt}) \rightarrow \text{Syn}_{\mathcal{B}}(\mathcal{X})$ (see [33, Corollary 2.8]). It follows

that we have a commutative diagram

$$\begin{array}{ccc} \mathcal{P}_\Sigma(\mathcal{B}; \text{Spt}) & \xrightarrow{\tau_{\geq 0}^{\text{pw}}} & \mathcal{P}_\Sigma(\mathcal{B}; \text{Spt})_{\geq 0} \\ L \downarrow & & \downarrow L \\ \text{Syn}_{\mathcal{B}}(\mathcal{X}) & \xrightarrow{\tau_{\geq 0}} & \text{Syn}_{\mathcal{B}}(\mathcal{X})_{\geq 0}, \end{array}$$

where $\tau_{\geq 0}^{\text{pw}} : \mathcal{P}_\Sigma(\mathcal{B}; \text{Spt}) \rightarrow \mathcal{P}_\Sigma(\mathcal{B}; \text{Spt})_{\geq 0}$ denotes the pointwise connective cover functor.

Remark 6.2.7. The Yoneda lemma gives a natural equivalence of spectra

$$\underline{\text{Map}}_{\text{Syn}_{\mathcal{B}}(\mathcal{X})}(\nu(B), S) \simeq S(B)$$

for $B \in \mathcal{B}$ and $S \in \text{Syn}_{\mathcal{B}}(\mathcal{X})$. It follows that the objects $\nu_{\mathcal{B}}(B_i[2n]) \in \text{Syn}_{\mathcal{B}}(\mathcal{X})$ for $i \in \mathbf{I}$ and $n \in \mathbb{Z}$ form a set of stable compact generators (cf. [33, Lemma 3.4]).

Construction 6.2.8. For each $B \in \mathcal{B}$, we have a canonical diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \tau_{\geq 4} Y_{\mathcal{B}}(B) & \longrightarrow & \tau_{\geq 2} Y_{\mathcal{B}}(B) & \longrightarrow & \tau_{\geq 0} Y_{\mathcal{B}}(B) & \longrightarrow & \tau_{\geq -2} Y_{\mathcal{B}}(B) & \longrightarrow & \tau_{\geq -4} Y_{\mathcal{B}}(B) & \longrightarrow & \cdots \\ & & \downarrow \sim & & \downarrow \sim & & \downarrow \sim & & \downarrow \sim & & \downarrow \sim & & \\ \cdots & \longrightarrow & \nu_{\mathcal{B}}(B[-4])[4] & \longrightarrow & \nu_{\mathcal{B}}(B[-2])[2] & \longrightarrow & \nu_{\mathcal{B}}(B) & \longrightarrow & \nu_{\mathcal{B}}(B[2])[-2] & \longrightarrow & \nu_{\mathcal{B}}(B[4])[-4] & \longrightarrow & \cdots \end{array}$$

in $\text{Syn}_{\mathcal{B}}(\mathcal{X})$, the two equivalent rows corepresenting a functor from $\text{Syn}_{\mathcal{B}}(\mathcal{X})$ to Fil . Specializing to $B = B_i$ for $i \in \mathbf{I}$, we get a functor

$$u_{\mathcal{B}} : \text{Syn}_{\mathcal{B}}(\mathcal{X}) \rightarrow \text{Fil}^{\mathbf{I}}.$$

We then set

$$\text{fil}_{\mathcal{B}\text{-pev}}^* := u_{\mathcal{B}} \circ \nu_{\mathcal{B}} : \mathcal{X} \rightarrow \text{Fil}^{\mathbf{I}},$$

and refer to this as the $\mathcal{B}$ -perfect even filtration. It follows from the compact generation statement in Remark 6.2.7 that the functor $u_{\mathcal{B}}$ is strongly algebraic, i.e. is conservative and preserves limits and colimits. Its left adjoint $f_{\mathcal{B}} : \text{Fil}^{\mathbf{I}} \rightarrow \text{Syn}_{\mathcal{B}}(\mathcal{X})$ sends $\mathbb{S}(-i)\{-n\}[2n]$ to $\nu(B_i[2n])$ for each $i \in \mathbf{I}$ and $n \in \mathbb{Z}$.

Remark 6.2.9. It may be disorienting that the above notion of $\mathcal{B}$ -perfect even filtration produces just a collection of filtered spectra, rather than, say, a filtered object of $\mathcal{X}$. In more specific cases of our general context, there are natural promotions of the construction that one may work with instead, coming from factorizations of $u_{\mathcal{B}}$; we leave further details and examples to Remark 6.4.7. Of course, its true home is $\text{Syn}_{\mathcal{B}}(\mathcal{X})$ itself.

Remark 6.2.10. Applying the Yoneda lemma as in Remark 6.2.7, we get the formula

$$u_{\mathcal{B}}(S)_i^n \simeq \underline{\text{Map}}_{\text{Syn}_{\mathcal{B}}}(\nu(B_i[2n])[-2n], S) \simeq \underline{\text{Map}}_{\text{Syn}_{\mathcal{B}}}(\nu(B_i[2n]), S)[2n] \simeq S(B_i[2n])[2n]$$

for $S \in \text{Syn}_{\mathcal{B}}(\mathcal{X})$, $i \in \mathbf{I}$, and $n \in \mathbb{Z}$.

Remark 6.2.11. For any $S \in \text{Syn}_{\mathcal{B}}(\mathcal{X})$ and $m \in \mathbb{Z}$, we have another object $S\{m\} \in \text{Syn}_{\mathcal{B}}(\mathcal{X})$ given by the formula $S\{m\}(B) := S(B[2m])[2m]$. This defines an automorphism of $\{m\} : \text{Syn}_{\mathcal{B}}(\mathcal{X}) \rightarrow \text{Syn}_{\mathcal{B}}(\mathcal{X})$, and by Remark 6.2.10, we have a natural equivalence $u_{\mathcal{B}}(S\{m\})^n \simeq u_{\mathcal{B}}(S)^{n+m}$ for each $n \in \mathbb{Z}$.

Example 6.2.12. For $X \in \mathcal{X}$ and $m \in \mathbb{Z}$, we have natural equivalences

$$Y_{\mathcal{B}}(X)\{m\} \simeq Y_{\mathcal{B}}(X[-2m])[2m] \simeq Y_{\mathcal{B}}(X), \quad \nu_{\mathcal{B}}(X)\{m\} \simeq \nu_{\mathcal{B}}(X[-2m])[2m].$$

Note that the automorphism $\{m\}$ carries the canonical t-structure $(\text{Syn}_{\mathcal{B}}(\mathcal{X})_{\geq 0}, \text{Syn}_{\mathcal{B}}(\mathcal{X})_{\leq 0})$ to the shift $(\text{Syn}_{\mathcal{B}}(\mathcal{X})_{\geq 2m}, \text{Syn}_{\mathcal{B}}(\mathcal{X})_{\leq 2m})$.

As in [33, Proposition 2.23], the colimit of the $\mathcal{B}$ -perfect even filtration has a straightforward description.

Notation 6.2.13. Let $\rho_{\mathcal{B}} : \mathcal{X} \rightarrow \text{Spt}^{\mathbf{I}}$ denote the functor corepresented by the set of generators $\mathcal{B}$.

Proposition 6.2.14. 1. *The following square commutes:*

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{Y_{\mathcal{B}}} & \mathrm{Syn}_{\mathcal{B}}(\mathcal{X}) \\ \rho_{\mathcal{B}} \downarrow & & \downarrow u_{\mathcal{B}} \\ \mathrm{Spt}^I & \xrightarrow{\delta} & \mathrm{Fil}^I. \end{array}$$

2. *For a coconnective object $S \in \mathrm{Syn}_{\mathcal{B}}(\mathcal{X})_{\leq 0}$, we have $\mathrm{colim}(u_{\mathcal{B}}(S)) \simeq 0$;*

3. *For $X \in \mathcal{X}$, we have a natural equivalence $\mathrm{colim}(\mathrm{fil}_{\mathcal{B}\text{-pev}}^*(X)) \simeq \rho_{\mathcal{B}}(X)$ in Spt^I .*

Proof. For 1 and 2, we use the formula of Remark 6.2.10. For $X \in \mathcal{X}$, $i \in I$, and $n \in \mathbb{Z}$, we have

$$Y_{\mathcal{B}}(X)(B_i[2n])[2n] = \underline{\mathrm{Map}}_{\mathcal{X}}(B_i[2n], X)[2n] \simeq \underline{\mathrm{Map}}_{\mathcal{X}}(B_i, X),$$

which gives 1. To prove 2, we observe that the coconnectivity of S implies that $S(B_i[2n])[2n]$ is a $2n$ -coconnective spectrum for each $i \in I$ and $n \in \mathbb{Z}$, whence the colimit over $n \in \mathbb{Z}^{\mathrm{op}}$ vanishes. Finally, it follows from 2 that, for $X \in \mathcal{X}$, the connective cover map $c: \nu(X) \rightarrow Y(X)$ becomes an equivalence upon applying the composition $\mathrm{colim} \circ u_{\mathcal{B}}$; combining this with 1 gives 3. $\square$

Next we note, as in [33, Lemma 2.36], that for those objects of $\mathcal{X}$ that are “even”, in a sense dual to the objects of $\mathcal{B}$, the $\mathcal{B}$ -perfect even filtration itself has a straightforward description.

Definition 6.2.15. We say that an object $E \in \mathcal{X}$ is $\mathcal{B}$ -even if, for each $B \in \mathcal{B}$, we have $\mathrm{Ext}_{\mathcal{X}}^1(B, E) \simeq 0$. This is equivalent to $\mathrm{Ext}_{\mathcal{X}}^*(B_i, E) = \pi_{-*}\underline{\mathrm{Map}}_{\mathcal{X}}(B_i, E)$ being concentrated in even degrees for each $i \in I$. We let $\mathcal{X}^{\mathcal{B}\text{-ev}} \subseteq \mathcal{X}$ denote the full subcategory spanned by the $\mathcal{B}$ -even objects.

In the case of Example 6.2.2, in particular when $\mathcal{X} = \mathrm{Spt}$ and $\mathcal{B} = \{\mathbb{S}\}$, we allow ourselves to omit $\mathcal{B}$ from this terminology and notation.

Proposition 6.2.16. *For $E \in \mathcal{X}^{\mathcal{B}\text{-ev}}$, we have a natural equivalence $\mathrm{fil}_{\mathcal{B}\text{-pev}}^*(E) \simeq \tau_{\geq 2*}(\rho_{\mathcal{B}}(E))$ in Fil^I .*

Proof. This amounts to the connective cover map $c: \nu_{\mathcal{B}}(E) \rightarrow Y_{\mathcal{B}}(E)$ in fact being a pointwise connective cover, which is equivalent to the pointwise connective cover

$$\tau_{\geq 0}^{\mathrm{pw}}(Y_{\mathcal{B}}(E)): \mathcal{B}^{\mathrm{op}} \rightarrow \mathrm{Spt}, \quad B \mapsto \tau_{\geq 0}\underline{\mathrm{Map}}_{\mathcal{X}}(B, E)$$

being an object of $\mathrm{Syn}_{\mathcal{B}}(\mathcal{X})$, meaning that it preserves fiber/cofiber sequences $B' \rightarrow B \rightarrow B''$ in $\mathcal{B}$ that remain fiber/cofiber sequences in $\mathcal{X}$. This holds by the vanishing of $\pi_{-1}\underline{\mathrm{Map}}_{\mathcal{X}}(B'', E) = \mathrm{Ext}_{\mathcal{X}}^1(B'', E)$. $\square$

We close this section by reviewing a key suite of definitions and results from [33], which provides a way to bootstrap from Proposition 6.2.16 to calculate the $\mathcal{B}$ -perfect even filtration for a larger class of objects of $\mathcal{X}$.

Definition 6.2.17. We say that an object $X \in \mathcal{X}$ is:

1. $\mathcal{B}$ -homologically even if the homotopy groups of $Y_{\mathcal{B}}(X)$ (with respect to the canonical t-structure on $\mathrm{Syn}_{\mathcal{B}}(\mathcal{X})$) are concentrated in even degrees (cf. [33, Definition 2.16]);
2. $\mathcal{B}$ -flat if it is a filtered colimit of objects of $\mathcal{B}$ (cf. [33, Definition 4.4]).

Remark 6.2.18. $\mathcal{B}$ -flatness implies $\mathcal{B}$ -homological evenness. This reduces to the objects of $\mathcal{B}$ being $\mathcal{B}$ -homologically even, for which see [33, Lemma 2.18].

Definition 6.2.19. Let $X \in \mathcal{X}$.

1. We say that a map $\phi: X \rightarrow E$ in $\mathcal{X}$ is a $\mathcal{B}$ -even envelope if E is $\mathcal{B}$ -even, ϕ is an $\mathcal{X}^{\mathcal{B}\text{-ev}}$ -cocovering (Definition 1.2.21), and the cofiber of ϕ is $\mathcal{B}$ -flat (cf. [33, Definition 4.7]).
2. We say that a cosemisimplicial object $E^{\bullet}: \Delta_s \rightarrow \mathcal{X}_{X/}$ is a $\mathcal{B}$ -even resolution if, for each nonnegative integer n , the matching map $E^{\bullet}[\partial\Delta_s^n] \rightarrow E^n$ (Notation 1.2.15) is a $\mathcal{B}$ -even envelope (cf. [33, Proof of Theorem 5.6]).

Proposition 6.2.20. *Let $X \in \mathcal{X}$. Then:*

1. *There exists a $\mathcal{B}$ -even envelope $\phi : X \rightarrow E$ in $\mathcal{X}$. If $\mathcal{B}$ is projective (Remark 6.2.4) and X is connective (with respect to the t -structure defined by $\mathcal{B}$), there exists one whose cofiber is 2-connective.*
2. *There exists a $\mathcal{B}$ -even resolution $E^\bullet : \Delta_s \rightarrow \mathcal{X}_{X/}$.*

Proof. Statement 2 follows immediately from statement 1, using Proposition 1.2.23. Statement 1 is an adaptation of [33, Proposition 4.11]; let us review the construction. We set $X_0 := X$ and then inductively define for each positive integer n a map $X_{n-1} \rightarrow X_n$ by forming a cofiber sequence

$$\bigoplus_{i,k,x} B_i[2k-1] \rightarrow X_{n-1} \rightarrow X_n,$$

where the sum is indexed over $i \in \mathbf{I}$, $k \in \mathbb{Z}$, and $x \in \text{Ext}_{\mathcal{X}}^{1-2k}(B_i, X) = \pi_0 \text{Map}_{\mathcal{X}}(B_i[2k-1], X)$, and the first map is given on the factor indexed by (i, k, x) by a representative of x . Then we may take $E := \text{colim } X_n$. $\square$

Proposition 6.2.21. *Let $X \in \mathcal{X}$ be $\mathcal{B}$ -homologically even and let $E^\bullet : \Delta_s \rightarrow \mathcal{X}_{X/}$ be a $\mathcal{B}$ -even resolution. Then the induced map*

$$\text{gr}_{\mathcal{B}\text{-pev}}^*(X) \rightarrow \lim_{\Delta_s} \text{gr}_{\mathcal{B}\text{-pev}}^*(E^\bullet)$$

is an equivalence.

Proof. See [33, Proposition 5.5 and Theorem 5.6]. $\square$

§6.3. The even filtration. We now reformulate Proposition 6.2.21 in terms of the following variant of the $\mathcal{B}$ -perfect even filtration, which is a direct analogue of (though certainly not the same as) the even filtration functor on $\mathbb{E}_\infty$ -rings defined in [17].

Construction 6.3.1. Recall the restricted Yoneda functor $\nu_{\mathcal{B}} : \mathcal{X} \rightarrow \text{Syn}_{\mathcal{B}}(\mathcal{X})$ (Construction 6.2.5) and the full subcategory of $\mathcal{B}$ -even objects $\mathcal{X}^{\mathcal{B}\text{-ev}} \subseteq \mathcal{X}$ (Definition 6.2.15). We may restrict $\nu_{\mathcal{B}}$ along the inclusion $\mathcal{X}^{\mathcal{B}\text{-ev}} \hookrightarrow \mathcal{X}$ and then right Kan extend back along this inclusion (the existence of the right Kan extension is justified in Remark 6.3.2 below); we denote the resulting functor by

$$e_{\mathcal{B}} : \mathcal{X} \rightarrow \text{Syn}_{\mathcal{B}}(\mathcal{X}), \quad X \mapsto \lim_{X \rightarrow E} \nu_{\mathcal{B}}(E),$$

the limit here indexed by the ∞ -category of maps $X \rightarrow E$ in $\mathcal{X}$ where E is $\mathcal{B}$ -even.

Parallel to the $\mathcal{B}$ -perfect even filtration functor $\text{fil}_{\mathcal{B}\text{-pev}}^*$, defined as the composition of $\nu_{\mathcal{B}}$ with the functor $u_{\mathcal{B}} : \text{Syn}_{\mathcal{B}} \rightarrow \text{Fil}^{\mathbf{I}}$ (Construction 6.2.8), we define

$$\text{fil}_{\mathcal{B}\text{-ev}}^* := u_{\mathcal{B}} \circ e_{\mathcal{B}} : \mathcal{X} \rightarrow \text{Fil}^{\mathbf{I}}$$

and refer to this as the $\mathcal{B}$ -even filtration functor. Since $u_{\mathcal{B}}$ preserves limits, this is equivalently obtained by restricting and then right Kan extending $\text{fil}_{\mathcal{B}\text{-pev}}^*$ along the inclusion $\mathcal{X}^{\mathcal{B}\text{-ev}} \hookrightarrow \mathcal{X}$. Proposition 6.2.16 identifies the restriction with the composition $\tau_{\geq 2*} \circ \rho_{\mathcal{B}}$, where $\rho_{\mathcal{B}} : \mathcal{X} \rightarrow \text{Spt}^{\mathbf{I}}$ is the functor corepresented by the generating set $\mathcal{B}$ and $\tau_{\geq 2*} : \text{Spt}^{\mathbf{I}} \rightarrow \text{Fil}^{\mathbf{I}}$ is the double speed Postnikov filtration functor. Thus, for $X \in \mathcal{X}$, we have a natural equivalence

$$\text{fil}_{\mathcal{B}\text{-ev}}^*(X) \simeq \lim_{X \rightarrow E} \tau_{\geq 2*}(\rho_{\mathcal{B}}(E)),$$

the limit again indexed by the ∞ -category of maps $X \rightarrow E$ in $\mathcal{X}$ where E is $\mathcal{B}$ -even.

Finally, let us write

$$\alpha : \nu_{\mathcal{B}} \rightarrow e_{\mathcal{B}}, \quad \bar{\alpha} : \text{fil}_{\mathcal{B}\text{-pev}}^* \rightarrow \text{fil}_{\mathcal{B}\text{-ev}}^*$$

for the unit transformations associated to the adjunction between restriction and right Kan extension (satisfying $\bar{\alpha} \simeq u_{\mathcal{B}}(\alpha)$).

Remark 6.3.2. The right Kan extensions in Construction 6.3.1 exist because $\mathcal{X}^{\mathcal{B}\text{-ev}}$ is an accessible subcategory of $\mathcal{X}$: using [28, Tag 06U9], this follows from $\mathcal{X}^{\mathcal{B}\text{-ev}}$ being the vanishing locus of a set of

accessible functors from $\mathcal{X}$ to $\mathbf{Ab}$. Accessibility implies that the limits appearing in Construction 6.3.1 are equivalent to small limits; more precisely, $e_{\mathcal{B}}$, for example, is given by the composition

$$\mathcal{X} \xrightarrow{j_{\mathcal{X}|\mathcal{X}^{\mathcal{B}\text{-ev}}}^\vee} \mathcal{P}^\vee(\mathcal{X}^{\mathcal{B}\text{-ev}}) \xrightarrow{\Gamma_{\nu_{\mathcal{B}}}} \mathrm{Syn}_{\mathcal{B}}(\mathcal{X})$$

(see Convention 0.3.10.c).

Remark 6.3.3. The description of $e_{\mathcal{B}}$ in Remark 6.3.2 above makes clear the following property: any cone in $\mathcal{X}$ that is carried to a limit diagram by the restricted Yoneda functor $j_{\mathcal{X}|\mathcal{X}^{\mathcal{B}\text{-ev}}}^\vee$ is also carried to a limit diagram by $e_{\mathcal{B}}$. (This is a general feature of right Kan extensions.)

Remark 6.3.4. For any $X \in \mathcal{X}$, the $\mathcal{B}$ -even filtration $\mathrm{fil}_{\mathcal{B}\text{-ev}}^*(X) \in \mathrm{Fil}^I$ is complete, since the Postnikov filtration is complete and completeness is closed under limits. On the other hand, the $\mathcal{B}$ -perfect even filtration is not generally complete.

Proposition 6.3.5. *Let $X \in \mathcal{X}$ be $\mathcal{B}$ -homologically even. Then the map $\bar{\alpha} : \mathrm{fil}_{\mathcal{B}\text{-pev}}^*(X) \rightarrow \mathrm{fil}_{\mathcal{B}\text{-ev}}^*(X)$ of Construction 6.3.1 exhibits $\mathrm{fil}_{\mathcal{B}\text{-ev}}^*(X)$ as the completion of $\mathrm{fil}_{\mathcal{B}\text{-pev}}^*(X)$.*

Proof. Since $\mathrm{fil}_{\mathcal{B}\text{-ev}}^*(X)$ is complete (Remark 6.3.3), it suffices to show that $\bar{\alpha}$ induces an equivalence on associated graded objects. Choose a $\mathcal{B}$ -even resolution $E^\bullet : \Delta_s^{\mathrm{op}} \rightarrow \mathcal{X}_{X/}$ (which exists by Proposition 6.2.20.2), and consider the induced commutative square

$$\begin{array}{ccc} \mathrm{gr}_{\mathcal{B}\text{-pev}}^*(X) & \xrightarrow{\bar{\alpha}} & \mathrm{gr}_{\mathcal{B}\text{-ev}}^*(X) \\ \downarrow & & \downarrow \\ \lim_{\Delta_s} \mathrm{gr}_{\mathcal{B}\text{-pev}}^*(E^\bullet) & \xrightarrow{\bar{\alpha}} & \lim_{\Delta_s} \mathrm{gr}_{\mathcal{B}\text{-ev}}^*(E^\bullet). \end{array}$$

Since E^n is $\mathcal{B}$ -even for each nonnegative integer n , the lower horizontal map is an equivalence. The right vertical map is an equivalence by Remark 6.3.3, because a $\mathcal{B}$ -even resolution is, by definition, in particular an $\mathcal{X}^{\mathcal{B}\text{-ev}}$ -cohypercoring (Definition 1.2.21), hence carried to a limit diagram by the restricted Yoneda functor $j_{\mathcal{X}|\mathcal{X}^{\mathcal{B}\text{-ev}}}^\vee : \mathcal{X} \rightarrow \mathrm{Fun}(\mathcal{X}^{\mathcal{B}\text{-ev}}, \mathbf{Spc})^{\mathrm{op}}$. The left vertical map is an equivalence by Proposition 6.2.21. We deduce the the upper horizontal map is an equivalence, as desired. $\square$

Remark 6.3.6. Proposition 6.3.5 does not generalize to all $X \in \mathcal{X}$. We give the simplest example to illustrate. Let R be an even $\mathbb{E}_1$ -ring, suppose that $\mathcal{X} = \mathrm{Mod}_R$ and $\mathcal{B} = \{R\}$ (we will drop $\mathcal{B}$ and $\mathcal{B}$ from the notation here), and take $X = R[1]$. Then we have

$$\mathrm{fil}_{\mathrm{pev}}^*(R[1]) \simeq \tau_{\geq 2*}(R[1]) \simeq \tau_{\geq 2*}(R)[1],$$

both equivalences using that R is even: the first is by [33, Proposition 2.40] and the second is elementary. On the other hand, observe that the zero map $R[1] \rightarrow 0$ is a $\mathrm{Mod}_R^{\mathrm{ev}}$ -cocovering. Thus, applying Lemma 6.3.14 below to the fiber sequence $R[1] \rightarrow 0 \rightarrow R[2]$ gives

$$\mathrm{fil}_{\mathrm{ev}}^*(R[1]) \simeq \mathrm{fil}_{\mathrm{ev}}^*(R[2])[-1] \simeq \tau_{\geq 2*}(R[2])[-1] \simeq \tau_{\geq 2(*-1)}(R)[1].$$

To further our understanding of the relationship between the $\mathcal{B}$ -perfect even and $\mathcal{B}$ -even filtrations, let us investigate the colimit of the latter.

Construction 6.3.7. In the same vein as Construction 6.3.1, we denote the right Kan extension of the inclusion functor $\mathcal{X}^{\mathcal{B}\text{-ev}} \hookrightarrow \mathcal{X}$ along itself by

$$(-)_{\mathcal{B}\text{-ev}}^\wedge : \mathcal{X} \rightarrow \mathcal{X}, \quad X \mapsto \lim_{X \rightarrow E} E.$$

Equivalently, as in Remark 6.3.2, this is the composition

$$\mathcal{X} \xrightarrow{j_{\mathcal{X}|\mathcal{X}^{\mathcal{B}\text{-ev}}}^\vee} \mathcal{P}^\vee(\mathcal{X}^{\mathcal{B}\text{-ev}}) \xrightarrow{\Gamma_{\mathcal{X}|\mathcal{X}^{\mathcal{B}\text{-ev}}}} \mathrm{Syn}_{\mathcal{B}}(\mathcal{X}).$$

We write $\beta : \mathrm{id}_{\mathcal{X}} \rightarrow (-)_{\mathcal{B}\text{-ev}}^\wedge$ for the unit transformation. Following Definition 1.2.5, we say that an object $X \in \mathcal{X}$ is $\mathcal{B}$ -even corealized if $\beta : X \rightarrow X_{\mathcal{B}\text{-ev}}^\wedge$ is an equivalence.

Remark 6.3.8. The spectral Yoneda functor $Y_{\mathcal{B}} : \mathcal{X} \rightarrow \text{Syn}_{\mathcal{B}}(\mathcal{X})$ (Construction 6.2.5) preserves limits, so restricting and right Kan extending it along the inclusion $\mathcal{X}^{\mathcal{B}\text{-ev}} \hookrightarrow \mathcal{X}$ is equivalent to forming the composition

$$Y_{\mathcal{B}} \circ (-)^{\wedge}_{\mathcal{B}\text{-ev}} : \mathcal{X} \rightarrow \text{Syn}_{\mathcal{B}}(\mathcal{X}), \quad X \mapsto Y_{\mathcal{B}}(X^{\wedge}_{\mathcal{B}\text{-ev}}) \simeq \lim_{X \rightarrow E} Y_{\mathcal{B}}(E).$$

Hence, the connective cover transformation $c : \nu_{\mathcal{B}} \rightarrow Y_{\mathcal{B}}$ induces a transformation $d : e_{\mathcal{B}} \rightarrow Y_{\mathcal{B}} \circ (-)^{\wedge}_{\mathcal{B}\text{-ev}}$, and we have the following natural commutative squares for $X \in \mathcal{X}$:

$$\begin{array}{ccc} \nu_{\mathcal{B}}(X) & \xrightarrow{\alpha} & e_{\mathcal{B}}(X) \\ c \downarrow & & \downarrow d \\ Y_{\mathcal{B}}(X) & \xrightarrow{\beta} & Y_{\mathcal{B}}(X^{\wedge}_{\mathcal{B}\text{-ev}}), \end{array} \quad \begin{array}{ccc} \text{fil}_{\mathcal{B}\text{-pev}}^*(X) & \xrightarrow{\bar{\alpha}} & \text{fil}_{\mathcal{B}\text{-ev}}^*(X) \\ \bar{c} \downarrow & & \downarrow \bar{d} \\ \delta(\rho_{\mathcal{B}}(X)) & \xrightarrow{\beta} & \delta(\rho_{\mathcal{B}}(X^{\wedge}_{\mathcal{B}\text{-ev}})); \end{array}$$

the left is in $\text{Syn}_{\mathcal{B}}(\mathcal{X})$, and the right, which is in Fil^I , is obtained by applying $u_{\mathcal{B}}$ to the left and then applying Proposition 6.2.14.1 to the bottom row. The other parts of Proposition 6.2.14 tell us that $\bar{c}$ is the adjunct of an equivalence $\text{colim}(\text{fil}_{\mathcal{B}\text{-pev}}^*(X)) \xrightarrow{\sim} \rho_{\mathcal{B}}(X)$, as c has (-1) -coconnective cofiber. Since coconnectivity is closed under limits, d also has (-1) -coconnective cofiber, and hence the adjunct of $\bar{d}$ is equivalence too:

$$\text{colim}(\text{fil}_{\mathcal{B}\text{-ev}}^*(X)) \xrightarrow{\sim} \rho_{\mathcal{B}}(X^{\wedge}_{\mathcal{B}\text{-ev}}).$$

Since $\rho_{\mathcal{B}}$ is conservative, this has the following consequence: if $\bar{\alpha}$ is an equivalence (which is equivalent to α being so), then $\beta : X \rightarrow X^{\wedge}_{\mathcal{B}\text{-ev}}$ is an equivalence, i.e. X is $\mathcal{B}$ -even corealized.

Construction 6.3.9. We define functors $e_{\mathcal{B}}^+ : \mathcal{X} \rightarrow \text{Syn}_{\mathcal{B}}(\mathcal{X})$ and $\text{fil}_{\mathcal{B}\text{-ev}+}^* : \mathcal{X} \rightarrow \text{Fil}^I$ by forming cartesian squares

$$\begin{array}{ccc} e_{\mathcal{B}}^+(X) & \xrightarrow{\beta'} & e_{\mathcal{B}}(X) \\ d^+ \downarrow & & \downarrow d \\ Y_{\mathcal{B}}(X) & \xrightarrow{\beta} & Y_{\mathcal{B}}(X^{\wedge}_{\mathcal{B}\text{-ev}}), \end{array} \quad \begin{array}{ccc} \text{fil}_{\mathcal{B}\text{-ev}+}^*(X) & \xrightarrow{\bar{\beta}'} & \text{fil}_{\mathcal{B}\text{-ev}}^*(X) \\ \bar{d}^+ \downarrow & & \downarrow \bar{d} \\ \delta\rho_{\mathcal{B}}(X) & \xrightarrow{\beta} & \delta\rho_{\mathcal{B}}(X^{\wedge}_{\mathcal{B}\text{-ev}}) \end{array}$$

for $X \in \mathcal{X}$. Under the equivalence $u_{\mathcal{B}} \circ Y_{\mathcal{B}} \simeq \delta \circ \rho_{\mathcal{B}}$ of Proposition 6.2.14.1, the right square identifies with the image of the left square under $u_{\mathcal{B}} : \text{Syn}_{\mathcal{B}}(\mathcal{X}) \rightarrow \text{Fil}^I$, and hence we have $\text{fil}_{\mathcal{B}\text{-ev}+}^* \simeq u_{\mathcal{B}} \circ e_{\mathcal{B}}^+$. The map $\bar{d}^+$ is the adjunct of an equivalence

$$\text{colim}(\text{fil}_{\mathcal{B}\text{-ev}+}^*(X)) \xrightarrow{\sim} \rho_{\mathcal{B}}(X);$$

this is equivalent to $\bar{d}^+$ inducing an equivalence on colimits, which is immediate from this being true of $\bar{d}$ (Remark 6.3.8) and the cartesianness of the right square above. The commutative squares of Remark 6.3.8 now define maps

$$\alpha^+ : \nu_{\mathcal{B}} \rightarrow e_{\mathcal{B}}^+, \quad \bar{\alpha}^+ : \text{fil}_{\mathcal{B}\text{-pev}}^* \rightarrow \text{fil}_{\mathcal{B}\text{-ev}+}^*,$$

with $\bar{\alpha}^+$ being equivalent to $u_{\mathcal{B}}(\alpha^+)$ and inducing an equivalence on colimits.

Proposition 6.3.10. *Let $X \in \mathcal{X}$. Then:*

1. *the projection map $\bar{\beta}' : \text{fil}_{\mathcal{B}\text{-ev}+}^*(X) \rightarrow \text{fil}_{\mathcal{B}\text{-ev}}^*(X)$ exhibits $\text{fil}_{\mathcal{B}\text{-ev}}^*(X)$ as the completion of $\text{fil}_{\mathcal{B}\text{-ev}+}^*(X)$;*
2. *$\text{fil}_{\mathcal{B}\text{-ev}+}^*(X)$ is complete if and only if X is $\mathcal{B}$ -even corealized.*

Proof. For both parts it is useful to have the criteria of Remark 6.1.8 in mind. Assertion 1 is then immediate from the cartesian square defining $\text{fil}_{\mathcal{B}\text{-ev}+}^*(X)$ and the fact that $\text{fil}_{\mathcal{B}\text{-ev}}^*(X)$ is complete (Remark 6.3.4), and it implies that $\text{fil}_{\mathcal{B}\text{-ev}+}^*(X)$ is complete if and only if $\bar{\beta}'$ induces an equivalence on colimits. The induced map on colimits identifies with $\rho_{\mathcal{B}}(\beta) : \rho_{\mathcal{B}}(X) \rightarrow \rho_{\mathcal{B}}(X^{\wedge}_{\mathcal{B}\text{-ev}})$, and since $\rho_{\mathcal{B}}$ is conservative, this map is an equivalence if and only if X is $\mathcal{B}$ -even corealized, giving 2. $\square$

Proposition 6.3.11. *Let $X \in \mathcal{X}$ be $\mathcal{B}$ -homologically even. Then:*

1. *the maps $\alpha^+ : \nu_{\mathcal{B}}(X) \rightarrow e_{\mathcal{B}}^+(X)$ and $\bar{\alpha}^+ : \text{fil}_{\mathcal{B}\text{-pev}}^*(X) \rightarrow \text{fil}_{\mathcal{B}\text{-ev}+}^*(X)$ of Construction 6.3.9 are equivalences;*

2. the maps $\alpha : \nu_{\mathcal{B}}(X) \rightarrow e_{\mathcal{B}}(X)$ and $\bar{\alpha} : \text{fil}_{\mathcal{B}\text{-pev}}^*(X) \rightarrow \text{fil}_{\mathcal{B}\text{-ev}}^*(X)$ of Construction 6.3.1 are equivalences if and only if X is $\mathcal{B}$ -even corealized.

Proof. Since $u_{\mathcal{B}} : \text{Syn}_{\mathcal{B}}(\mathcal{X}) \rightarrow \text{Fil}^I$ is conservative, α^+ (resp. α) is an equivalence if and only if $\bar{\alpha}^+$ (resp. $\bar{\alpha}$) is an equivalence. By Proposition 6.3.5 and Proposition 6.3.10.1, $\bar{\alpha}^+$ induces an equivalence on completions; since it induces an equivalence on colimits by construction, we deduce that it is an equivalence (Remark 6.1.8), proving 1. Then 2 follows from Proposition 6.3.10.2. $\square$

We note the following criterion for $\mathcal{B}$ -even corealization.

Proposition 6.3.12. *Assume that $\mathcal{B}$ is projective (Remark 6.2.4). Then any object $X \in \mathcal{X}$ that is right bounded with respect to the t -structure $(\mathcal{X}_{\geq 0}^{\mathcal{B}}, \mathcal{X}_{\leq 0}^{\mathcal{B}})$ is $\mathcal{B}$ -even corealized.*

In view of Proposition 6.3.11.2, Proposition 6.3.12 may be regarded as a generalization of [33, Theorem 8.3.2]. To prove it, we establish analogues for the $\mathcal{B}$ -even filtration of a few more techniques with the $\mathcal{B}$ -perfect even filtration from [33].

Lemma 6.3.13. *For $X \in \mathcal{X}$ and $n \in \mathbb{Z}$, we have natural equivalences*

$$e_{\mathcal{B}}(X[2n]) \simeq e_{\mathcal{B}}(X)\{-n\}[2n], \quad e_{\mathcal{B}}^+(X[2n]) \simeq e_{\mathcal{B}}^+(X)\{-n\}[2n],$$

$$\text{fil}_{\mathcal{B}\text{-ev}}^*(X[2n]) \simeq \text{fil}_{\mathcal{B}\text{-ev}}^{*-n}(X)[2n], \quad (X[2n])_{\mathcal{B}\text{-ev}}^{\wedge} \simeq X_{\mathcal{B}\text{-ev}}^{\wedge}[2n], \quad \text{fil}_{\mathcal{B}\text{-ev}^+}^*(X[2n]) \simeq \text{fil}_{\mathcal{B}\text{-ev}^+}^{*-n}(X)[2n].$$

Proof. The first two follow from Example 6.2.12, and applying $u_{\mathcal{B}}$ to these produces the third and fifth. Taking colimits of the third gives the fourth. $\square$

Lemma 6.3.14. *Let $X \xrightarrow{\phi} Y \xrightarrow{\psi} Z$ be a fiber sequence in $\mathcal{X}$, and assume that ϕ is an $\mathcal{X}^{\mathcal{B}\text{-ev}}$ -cocovering (Definition 1.2.21). Then this fiber sequence is preserved by the functors*

$$e_{\mathcal{B}}, e_{\mathcal{B}}^+ : \mathcal{X} \rightarrow \text{Syn}_{\mathcal{B}}(\mathcal{X}), \quad \text{fil}_{\mathcal{B}\text{-ev}}^*, \text{fil}_{\mathcal{B}\text{-ev}^+}^* : \mathcal{X} \rightarrow \text{Fil}^I.$$

Proof. It suffices to address the claim for the former pair of functors, since the latter pair is obtained from them by composition with $u_{\mathcal{B}} : \text{Syn}_{\mathcal{B}}(\mathcal{X}) \rightarrow \text{Fil}^I$. Furthermore, by the cartesian square defining $e_{\mathcal{B}}^+$ and the exactness of $Y_{\mathcal{B}}$, it suffices to address the claim for $e_{\mathcal{B}}$.

Recall from Remark 6.3.2 that $e_{\mathcal{B}}$ factors as the composition

$$\mathcal{X} \xrightarrow{j_{\mathcal{X}|\mathcal{X}^{\mathcal{B}\text{-ev}}}^{\vee}} \mathcal{P}^{\vee}(\mathcal{X}^{\mathcal{B}\text{-ev}})^{\text{op}} \xrightarrow{\Gamma_{\nu_{\mathcal{B}}}} \text{Syn}_{\mathcal{B}}(\mathcal{X}).$$

Let $\mathcal{P}_{\Sigma}^{\vee}(\mathcal{X}^{\mathcal{B}\text{-ev}}) \subseteq \mathcal{P}^{\vee}(\mathcal{X}^{\mathcal{B}\text{-ev}})$ be the full subcategory spanned by those copreshaves that preserve finite products; by [29, Proposition C.1.5.7], this is a prestable ∞ -category. For any $X \in \mathcal{X}$, the corepresentable functor $j_{\mathcal{X}|\mathcal{X}^{\mathcal{B}\text{-ev}}}^{\vee}(X) = \text{Map}_{\mathcal{X}}(X, -)$ lies in $\mathcal{P}_{\Sigma}^{\vee}(\mathcal{X}^{\mathcal{B}\text{-ev}})$. In addition, since $\nu_{\mathcal{B}} : \mathcal{X} \rightarrow \text{Syn}_{\mathcal{B}}(\mathcal{X})$, the functor $\Gamma_{\nu_{\mathcal{B}}}$ factors through a limit-preserving functor $\mathcal{P}_{\Sigma}^{\vee}(\mathcal{X}^{\mathcal{B}\text{-ev}})^{\text{op}} \rightarrow \text{Syn}_{\mathcal{B}}(\mathcal{X})$. Therefore, it suffices to show that the given fiber sequence in $\mathcal{X}$ is carried by $j_{\mathcal{X}|\mathcal{X}^{\mathcal{B}\text{-ev}}}^{\vee}$ to a fiber sequence in $\mathcal{P}_{\Sigma}^{\vee}(\mathcal{X}^{\mathcal{B}\text{-ev}})^{\text{op}}$, i.e. a cofiber sequence in $\mathcal{P}_{\Sigma}^{\vee}(\mathcal{X}^{\mathcal{B}\text{-ev}})$. It is evidently carried to a fiber sequence in $\mathcal{P}_{\Sigma}^{\vee}(\mathcal{X}^{\mathcal{B}\text{-ev}})$, and the assumption that ϕ is a $\mathcal{X}^{\mathcal{B}\text{-ev}}$ -cocovering gives us that this fiber sequence in $\mathcal{P}_{\Sigma}^{\vee}(\mathcal{X}^{\mathcal{B}\text{-ev}})$ is also a cofiber sequence. $\square$

Proposition 6.3.15. *Let $X \in \mathcal{X}$. Then:*

1. *there exist a sequence of objects $\{E^k\}_{k \geq 0}$ in $\mathcal{X}^{\mathcal{B}\text{-ev}}$ and a filtration*

$$\cdots \rightarrow D^2 \rightarrow D^1 \rightarrow D^0 = e_{\mathcal{B}}(X)$$

in $\text{Syn}_{\mathcal{B}}(\mathcal{X})$ with equivalences $\text{gr}^k(D) \simeq e_{\mathcal{B}}(E^k)[-k]$ for each nonnegative integer k , where $\text{gr}^k(D) := \text{cofib}(D^{k+1} \rightarrow D^k)$;

2. *if $\mathcal{B}$ is projective and X is connective (with respect to the t -structure defined by $\mathcal{B}$), there exist data E^* and D^* as in 1 where additionally E^k is $(2k)$ -connective for each nonnegative integer k .*

Proof. This comes from [33, Construction 5.2 and Proposition 5.3]. Using Proposition 6.2.20.1 inductively,

we may construct a commutative diagram

$$\begin{array}{ccccccc}
& & E^0 & \xrightarrow{d^0} & E^1 & \xrightarrow{d^1} & E^2 \xrightarrow{d^2} \dots \\
& \nearrow \phi^0 & \downarrow \psi^0 & \nearrow \phi^1 & \downarrow \psi^1 & \nearrow \phi^2 & \downarrow \psi^2 \nearrow \phi^3 \\
X = X^0 & & X^1 & & X^2 & & X^3
\end{array}$$

in $\mathcal{X}$, where, for each nonnegative integer k , the map ϕ^k is a $\mathcal{B}$ -even envelope and the map ψ^k is obtained by taking the cofiber of ϕ^k . Let us apply $e_{\mathcal{B}}$ to this diagram. By Lemma 6.3.14, we have induced fiber sequences $e_{\mathcal{B}}(X^k) \rightarrow e_{\mathcal{B}}(E^k) \rightarrow e_{\mathcal{B}}(X^{k+1})$, rotating to fiber sequences $e_{\mathcal{B}}(X^{k+1})[-1] \rightarrow e_{\mathcal{B}}(X^k) \rightarrow e_{\mathcal{B}}(E^k)$. Setting $D^k := e_{\mathcal{B}}(X^k)[-k]$, this gives data as in 1. If $\mathcal{B}$ is projective and X is connective, then Proposition 6.2.20.1 tells us that we may arrange in the above construction that E^k is $(2k)$ -connective for each $k \geq 0$. $\square$

Remark 6.3.16. Let $X \in \mathcal{X}$, and choose E^* and D^* as in Proposition 6.3.15.1. Let n be an integer, and let $v_{\mathcal{B}}^n : \text{Syn}_{\mathcal{B}}(\mathcal{X}) \rightarrow \text{Spt}^1$ be the composition

$$\text{Syn}_{\mathcal{B}}(\mathcal{X}) \xrightarrow{u_{\mathcal{B}}} \text{Fil} \xrightarrow{\text{gr}^n} \text{Spt}.$$

Applying $v_{\mathcal{B}}^n$ to D^* produces a nonnegatively filtered spectrum, with final term

$$v_{\mathcal{B}}^n(D^0) = v_{\mathcal{B}}^n(e_{\mathcal{B}}(X)) = \text{gr}_{\mathcal{B}\text{-ev}}^n(X),$$

and with associated graded pieces

$$\text{gr}^k(v_{\mathcal{B}}^n(D^*)) \simeq v_{\mathcal{B}}^n(\text{gr}^k(D^*)) \simeq v_{\mathcal{B}}^n(e_{\mathcal{B}}(E^k)[-k]) = \text{gr}_{\mathcal{B}\text{-ev}}^n(E^k)[-k] \simeq \pi_{2n}(\rho_{\mathcal{B}}(E^k))[2n - k].$$

for each nonnegative integer k . In other words, $\text{gr}_{\mathcal{B}\text{-ev}}^n(X)[-2n]$ is represented by a cochain complex of $\mathbb{I}$ -graded abelian groups of the following form:

$$\pi_{2n}(\rho_{\mathcal{B}}(E^0)) \rightarrow \pi_{2n}(\rho_{\mathcal{B}}(E^1)) \rightarrow \pi_{2n}(\rho_{\mathcal{B}}(E^2)) \rightarrow \dots$$

Corollary 6.3.17. Suppose that $\mathcal{B}$ is projective, and let $X \in \mathcal{X}_{\geq 0}^{\mathcal{B}}$. Then, for n a negative integer, $\text{gr}_{\mathcal{B}\text{-ev}}^n(X)$ vanishes, and for n a nonnegative integer, $\text{gr}_{\mathcal{B}\text{-ev}}^n(X)$ is concentrated in degrees $[n, 2n]$.

Proof. Choosing E^* and D^* as in Proposition 6.3.15.2, this follows from Remark 6.3.16. $\square$

We are now in position to imitate [33, Proof of Theorem 8.3].

Proof of Proposition 6.3.12. We will prove that $\text{fil}_{\mathcal{B}\text{-ev}+}^*(X)$ is complete for any right bounded $X \in \mathcal{X}$, which by Proposition 6.3.10.2 is equivalent to the desired claim. By Lemma 6.3.13, we are free to replace M by any even shift of X , so that it suffices to treat the case where X is connective.

Fix an integer $k \geq -2$. We will inductively show that $\lim(\text{fil}_{\mathcal{B}\text{-ev}+}^*(X)) \in \text{Spt}^1$ is (pointwise) k -connective for any connective $X \in \mathcal{X}$; in the end, this implies that $\lim(\text{fil}_{\mathcal{B}\text{-ev}+}^*(X)) \simeq 0$, as desired. For the base case $k = -2$, it suffices to show that $\text{fil}_{\mathcal{B}\text{-ev}+}^n(X)$ is (-1) -connective for each $n \geq 0$. We prove this by induction on n . For $n = 0$, we have $\text{fil}_{\mathcal{B}\text{-ev}+}^0(X) \simeq \rho_{\mathcal{B}}(X)$: this follows from the fact that $\text{colim}(\text{fil}_{\mathcal{B}\text{-ev}+}^*(X)) \simeq \rho_{\mathcal{B}}(X)$ (Construction 6.3.9), together with the fact from Corollary 6.3.17 that $\text{gr}_{\mathcal{B}\text{-ev}}^m(X) \simeq 0$ for $m < 0$. Suppose then that $n \geq 1$. By Proposition 6.3.10.1, we have a fiber sequence

$$\text{fil}_{\mathcal{B}\text{-ev}+}^n(X) \rightarrow \text{fil}_{\mathcal{B}\text{-ev}+}^{n-1}(X) \rightarrow \text{gr}_{\mathcal{B}\text{-ev}}^{n-1}(X).$$

By the inductive hypothesis, $\text{fil}_{\mathcal{B}\text{-ev}+}^{n-1}(X)$ is (-1) -connective, and by Corollary 6.3.17, $\text{gr}_{\mathcal{B}\text{-ev}}^{n-1}(X)$ is connective. Hence $\text{fil}_{\mathcal{B}\text{-ev}+}^n(X)$ is (-1) -connective, as desired.

Suppose now that $k \geq -1$. By Proposition 6.2.20.1, we may choose an $\mathcal{B}$ -even envelope $\phi : X \rightarrow E$ such that $C := \text{cofib}(\phi)$ is 2-connective. By Lemma 6.3.14, there is an induced fiber sequence

$$\text{fil}_{\mathcal{B}\text{-ev}+}^*(X) \rightarrow \text{fil}_{\mathcal{B}\text{-ev}+}^*(E) \rightarrow \text{fil}_{\mathcal{B}\text{-ev}+}^*(C).$$

Since E is $\mathcal{B}$ -even and hence $\mathcal{B}$ -even corealized, $\text{fil}_{\mathcal{B}\text{-ev}+}^*(E)$ is complete, so we get an equivalence $\text{fil}_{\mathcal{B}\text{-ev}+}^*(X) \simeq \text{fil}_{\mathcal{B}\text{-ev}+}^*(C)[-1]$, reducing us to showing that $\text{fil}_{\mathcal{B}\text{-ev}+}^*(C)$ is $(k+1)$ -connective. By Lemma 6.3.13,

we have

$$\mathrm{fil}_{\mathcal{B}\text{-ev}+}^*(C) \simeq \mathrm{fil}_{\mathcal{B}\text{-ev}+}^{*-1}(C[-2])[2] \implies \lim(\mathrm{fil}_{\mathcal{B}\text{-ev}+}^*(C)) \simeq \lim(\mathrm{fil}_{\mathcal{B}\text{-ev}+}^*(C[-2]))[2],$$

further reducing us to showing that $\lim(\mathrm{fil}_{\mathcal{B}\text{-ev}+}^*(C[-2]))$ is $(k-1)$ -connective. Since $C[-2]$ is connective, this follows from the inductive hypothesis. $\square$

§6.4. Conclusion. Let us finally state and prove our alternative characterizations of the ∞ -category $\mathrm{Syn}_{\mathcal{B}}(\mathcal{X})$ of Construction 6.2.5.

Notation 6.4.1. We let $A_{\mathcal{I}} \subseteq \mathrm{Fil}$ be the full subcategory generated under finite sums by the objects $\mathbb{S}\{-n\}[2n-m]$ for $n \in \mathbb{Z}$ and $m \in \mathbb{Z}_{\geq 0}$. Hence, $A_{\mathcal{I}}^I \subseteq \mathrm{Fil}^I$ is the full subcategory generated under finite sums by $\mathbb{S}(-i)\{-n\}[2n-m]$ for $i \in I$, $n \in \mathbb{Z}$, and $m \in \mathbb{Z}_{\geq 0}$. Observe that $\mathrm{Fil}^I|A_{\mathcal{I}}^I$ is a presented ∞ -category of the form considered in Ch. 4.

Theorem 6.4.2. *Suppose that the objects of $\mathcal{B}$ are $\mathcal{B}$ -even corealized (Construction 6.3.7). Then the commutative triangle (given by Proposition 6.2.16)*

$$\begin{array}{ccc} \mathcal{X}^{\mathcal{B}\text{-ev}} & \xrightarrow{\nu_{\mathcal{B}}} & \mathrm{Syn}_{\mathcal{B}}(\mathcal{X}) \\ & \searrow \tau_{\geq 2*} \circ \rho_{\mathcal{B}} & \downarrow u_{\mathcal{B}} \\ & & \mathrm{Fil}^I \end{array}$$

exhibits $\mathrm{Syn}_{\mathcal{B}}(\mathcal{X})$ as the $A_{\mathcal{I}}^I$ -theorization, algebraization, and strong algebraization of $\mathcal{X}^{\mathcal{B}\text{-ev}}$ over Fil^I .

Remark 6.4.3. It follows from Proposition 6.3.12 that the hypothesis of Theorem 6.4.2 is satisfied when $\mathcal{B}$ is projective (Remark 6.2.4). In particular, specializing to the case $\mathcal{X} = \mathrm{Spt}$ and $\mathcal{B} = \{\mathbb{S}\}$, where $\mathrm{Syn}_{\mathcal{B}}(\mathcal{X})$ is the ∞ -category of synthetic spectra $\mathrm{Syn}^{\mathrm{ev}} = \mathrm{Syn}_{\mathbb{S}}^{\mathrm{ev}}$, Theorem 6.4.2 implies Theorems 0.1.2 and 0.1.5 from the introduction.

Proof of Theorem 6.4.2. Since $u_{\mathcal{B}}$ is strongly algebraic, it suffices to prove the $A_{\mathcal{I}}^I$ -theorization claim (Remark 2.3.12). The functor $\tau_{\geq 2*} \circ \rho_{\mathcal{B}}$ is accessible and hence $A_{\mathcal{I}}^I$ -theorizable (Remark 6.3.2 and Example 2.2.17). Thus, the given commutative triangle factors through another,

$$\begin{array}{ccc} (\mathcal{X}^{\mathcal{B}\text{-ev}})^{A_{\mathcal{I}}^I\text{-th}/\mathrm{Fil}^I} & \xrightarrow{v} & \mathrm{Syn}_{\mathcal{B}}(\mathcal{X}) \\ & \searrow u & \downarrow u_{\mathcal{B}} \\ & & \mathrm{Fil}^I, \end{array}$$

where the upper left ∞ -category is the $A_{\mathcal{I}}^I$ -theorization of $\mathcal{X}^{\mathcal{B}\text{-ev}}$ over Fil^I and u denotes its structure map. Note that u is strongly algebraic, equivalently $(\mathcal{X}^{\mathcal{B}\text{-ev}})^{A_{\mathcal{I}}^I\text{-th}/\mathrm{Fil}^I}$ is stable; this follows from Proposition 4.3.6.

We would like to show that v is an equivalence. Let $f : \mathrm{Fil}^I \rightarrow (\mathcal{X}^{\mathcal{B}\text{-ev}})^{A_{\mathcal{I}}^I\text{-th}/\mathrm{Fil}^I}$ and $f_{\mathcal{B}} : \mathrm{Fil}^I \rightarrow \mathrm{Syn}_{\mathcal{B}}(\mathcal{X})$ denote the left adjoints of u and $u_{\mathcal{B}}$, respectively. By Proposition 2.2.26, it suffices to show that, for each $A \in A_{\mathcal{I}}^I$, the canonical map $f_{\mathcal{B}}(A) \rightarrow vf(A)$ in $\mathrm{Syn}_{\mathcal{B}}(\mathcal{X})$ is an equivalence. The exactness of the functors involved reduces us to showing this for $A = \mathbb{S}(-i)\{-n\}[2n]$ for $n \in \mathbb{Z}$. In this case, applying Proposition 2.2.27 and unwinding definitions identifies the map in question with the map

$$\alpha : \nu_{\mathcal{B}}(B_i[2n]) \rightarrow e_{\mathcal{B}}(B_i[2n])$$

of Construction 6.3.1. Since $B_i[2n] \in \mathcal{B}$ is $\mathcal{B}$ -homologically even by Remark 6.2.18 and $\mathcal{B}$ -even corealized by hypothesis, Proposition 6.3.11.2 tells us that the above map is an equivalence. $\square$

We now formulate a variant of Theorem 6.4.2, removing the $\mathcal{B}$ -even corealization hypothesis by adjoining a bit of extra data to $\mathcal{X}^{\mathcal{B}\text{-ev}}$.

Construction 6.4.4. We define $\mathcal{X}^{\mathcal{B}\text{-ev}+}$ to be the full subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{X})$ whose objects are those maps $X \rightarrow E$ in $\mathcal{X}$ where E is $\mathcal{B}$ -even. Note that we have a canonical fully faithful embedding $\epsilon : \mathcal{X}^{\mathcal{B}\text{-ev}} \hookrightarrow \mathcal{X}^{\mathcal{B}\text{-ev}+}$

with essential image consisting of the equivalences $X \xrightarrow{\sim} E$.

For an object $X \rightarrow E$ in $\mathcal{X}^{\mathcal{B}\text{-ev}+}$, we have a natural cospan

$$Y_{\mathcal{B}}(X) \rightarrow Y_{\mathcal{B}}(E) \leftarrow \nu_{\mathcal{B}}(E)$$

in $\text{Syn}_{\mathcal{B}}(\mathcal{X})$, the rightward map given by functoriality of Y and the leftward one by the connective cover transformation $c : \nu_{\mathcal{B}} \rightarrow Y_{\mathcal{B}}$. Applying Propositions 6.2.14 and 6.2.16, we find that image of this under $u_{\mathcal{B}}$ is the cospan

$$\delta(X') \rightarrow \delta(E') \leftarrow \tau_{\geq 2*}(E')$$

in Fil^I , where here $(X' \rightarrow E') := \rho_{\mathcal{B}}(X \rightarrow E) \in \text{Spt}^{\text{ev}+}$, the rightward map is given by the functoriality of δ , and the leftward map is given by the connective cover transformation $\tau_{\geq 2*} \rightarrow \delta$. We may take the fiber products of the above two cospans, defining functors

$$\begin{aligned} \nu_{\mathcal{B}}^+ : \mathcal{X}^{\mathcal{B}\text{-ev}+} &\rightarrow \text{Syn}_{\mathcal{B}}(\mathcal{X}), & (X \rightarrow E) &\mapsto Y(X) \times_{Y(E)} \nu(E), \\ \tau_{\geq 2*}^+ : \text{Spt}^{\text{ev}+} &\rightarrow \text{Fil}, & (X' \rightarrow E') &\mapsto \delta(X') \times_{\delta(E')} \tau_{\geq 2*}(E'). \end{aligned}$$

By construction, these fit into a commutative triangle

$$\begin{array}{ccc} \mathcal{X}^{\mathcal{B}\text{-ev}+} & \xrightarrow{\nu_{\mathcal{B}}^+} & \text{Syn}_{\mathcal{B}}(\mathcal{X}) \\ & \searrow \tau_{\geq 2*}^+ \circ \rho_{\mathcal{B}} & \downarrow u_{\mathcal{B}} \\ & & \text{Fil}^I. \end{array}$$

After precomposing with ϵ , this recovers the triangle of Theorem 6.4.2.

Theorem 6.4.5. *The commutative triangle of Construction 6.4.4 exhibits $\text{Syn}_{\mathcal{B}}(\mathcal{X})$ as the $A_{\mathcal{Z}}^I$ -theorization, algebraization, and strong algebraization of $\mathcal{X}^{\mathcal{B}\text{-ev}+}$ over Fil^I .*

Remark 6.4.6. The maneuver of passing from $\mathcal{X}^{\mathcal{B}\text{-ev}}$ to $\mathcal{X}^{\mathcal{B}\text{-ev}+}$ arose in discussions with Devalapurkar–Hahn–Senger and Lurie, in the context of the forthcoming joint work with the former group on “synthetic $\mathbb{E}_{\infty}$ -rings” advertised in Ch. 0, and a form of it will appear again in that work.

Proof of Theorem 6.4.5. [Similar to the proof of Theorem 6.4.2. Details to be included in future draft.] $\square$

Remark 6.4.7. In more specific cases, the functor $u_{\mathcal{B}} : \text{Syn}_{\mathcal{B}}(\mathcal{X}) \rightarrow \text{Fil}^I$, and hence the $\mathcal{B}$ -perfect even filtration functor $\text{fil}_{\mathcal{B}\text{-pev}}^* = u_{\mathcal{B}} \circ \nu_{\mathcal{B}}$, may factor through some other, naturally occurring, strongly algebraic functor $w : \mathcal{Z} \rightarrow \text{Fil}^I$ (perhaps where the relationship of the ∞ -category $\mathcal{Z}$ with $\mathcal{X}$ feels more canonical than that of Fil^I). In this situation, the conclusion of Theorem 6.4.2 (resp. Theorem 6.4.5) is equivalent to $\text{Syn}_{\mathcal{B}}(\mathcal{X})$ being the $\mathcal{A}$ -theorization, algebraization, and strong algebraization of $\mathcal{X}^{\mathcal{B}\text{-ev}}$ (resp. $\mathcal{X}^{\mathcal{B}\text{-ev}+}$) over $\mathcal{Z}$, where $\mathcal{A}$ is the essential image of $A_{\mathcal{Z}}^I$ under the left adjoint of w (Remark 3.2.13).

In fact, Theorems 6.4.2 and 6.4.5 provide a way to think about and construct such a factorization: it’s equivalent to a factorization through w of $\tau_{\geq 2*}^+ \circ \rho_{\mathcal{B}} : \mathcal{X}^{\mathcal{B}\text{-ev}+} \rightarrow \text{Fil}^I$, or just of $\tau_{\geq 2*} \circ \rho_{\mathcal{B}} : \mathcal{X}^{\mathcal{B}\text{-ev}} \rightarrow \text{Fil}^I$ when $\mathcal{B}$ is projective. Let us illustrate with a couple of examples:

1. Suppose that $\mathcal{B}$ is projective, so that we have an equivalence $\mathcal{X} \rightarrow \mathcal{P}_{\Sigma}(\mathcal{B}_0; \text{Spt})$, where $\mathcal{B}_0 \subseteq \mathcal{X}$ is the full subcategory generated under finite sums by the objects of $\mathcal{B}$ (Remark 6.2.4). Then the ∞ -category $\text{Fil}(\mathcal{X}) := \text{Fun}(\mathbb{Z}^{\text{op}}, \mathcal{X})$ of filtered objects of $\mathcal{X}$ identifies with the ∞ -category $\mathcal{P}_{\Sigma}(\mathcal{B}_0; \text{Fil}) \subseteq \text{Fun}(\mathcal{B}_0^{\text{op}}, \text{Fil})$ of finite product preserving functors from $\mathcal{B}_0$ to Fil , so that evaluation at the objects $B_i \in \mathcal{B}_0$ for $i \in I$ defines a strongly algebraic functor $w : \text{Fil}(\mathcal{X}) \rightarrow \text{Fil}^I$. It is clear from the definition of $u_{\mathcal{B}}$ how to factor it through w , namely by replacing $\mathcal{B}$ with $\mathcal{B}_0$ in the construction. The corresponding factorization of $\tau_{\geq 2*} \circ \rho_{\mathcal{B}}$ through w is given by the Postnikov filtration functor $\tau_{\geq 2*}^{\mathcal{B}} : \mathcal{X}^{\mathcal{B}\text{-ev}} \rightarrow \text{Fil}(\mathcal{X})$ associated to the t -structure on $\mathcal{X}$ determined by $\mathcal{B}$, equivalent to the pointwise t -structure on $\mathcal{P}_{\Sigma}(\mathcal{B}_0; \text{Spt})$.
2. Let’s begin again as in 1, assuming that the generating set $\mathcal{B}$ is projective, but this time discuss a mechanism by which $\mathcal{B}$ may be enlarged (losing projectivity in the process) to determine a potentially different instance of the constructions of this section. Example 6.2.3 will fall under the scope of this mechanism.

We suppose given a group Λ —whose group structure we'll write using additive notation, despite not being assumed abelian—together with an element $2 \in \Lambda$ and a group homomorphism $\Sigma^{(-)} : \Lambda \rightarrow \pi_0 \text{Aut}(\mathcal{X})$, such that, as the notation indicates, the image of $2 \in \Lambda$ under $\Sigma^{(-)}$ is the double suspension automorphism $\Sigma^2 = [2] : \mathcal{X} \rightarrow \mathcal{X}$ (and hence, for any integer n , the image of its n -th multiple $2n \in \Lambda$ is $\Sigma^{2n} = [2n]$). We then define our enlargement of $\mathcal{B}$ to be $\mathcal{B}_\Lambda := \{\Sigma^\lambda B_i\}_{(i,\lambda) \in \mathcal{I}_\Lambda}$, where $\mathcal{I}_\Lambda := \mathbb{I} \times \Lambda$; and we let $\mathcal{B}_\Lambda \subseteq \mathcal{X}$ denote the corresponding enlargement of $\mathcal{B}$, namely the full subcategory generated under extensions and retracts by the objects of $\mathcal{B}_\Lambda$, which then is already closed under even suspensions and desuspensions. (For notational convenience, we elide here the distinction between automorphisms of $\mathcal{X}$ and homotopy classes thereof; the set $\mathcal{B}_\Lambda$, and some related constructions below, depend on choices of representatives, but only up to equivalence.)

Now we may consider

$$u_{\mathcal{B}_\Lambda} : \text{Syn}_{\mathcal{B}_\Lambda}(\mathcal{X}) \rightarrow \text{Fil}^{\mathcal{I}_\Lambda}.$$

This functor does not necessarily factor through $\text{Fil}(\mathcal{X})$, as we had in 1, but there is the following replacement. Let us: equip Λ with the preorder where, for $\lambda, \mu \in \Lambda$, we have $\mu \leq \lambda$ if and only if $\Sigma^{-\mu+\lambda} \simeq \Sigma^{-\mu} \Sigma^\lambda : \mathcal{X} \rightarrow \mathcal{X}$ preserves the connective subcategory $\mathcal{X}_{\geq 0}$; set

$$\text{Fil}^\Lambda(\mathcal{X}) := \text{Fun}(\Lambda^{\text{op}}, \mathcal{X}),$$

where Λ is considered as a category by means of its preorder; refer to objects of $\text{Fil}^\Lambda(\mathcal{X})$ as Λ -*filtered objects* of $\mathcal{X}$; and denote them as sequences $X^* = \{X^\lambda\}_{\lambda \in \Lambda}$. Note that the preorder on Λ restricts to the usual one on the multiples of $2 \in \Lambda$, and that, for $\lambda, \mu, \xi \in \Lambda$, if $\xi \leq \mu$, then $\lambda + \xi \leq \lambda + \mu$. Thus, given $X^* \in \text{Fil}^\Lambda(\mathcal{X})$ and $\lambda \in \Lambda$, we have a filtered object $X^{\lambda+2*} \in \text{Fil}(\mathcal{X})$. This construction determines a strongly algebraic functor

$$w_\Lambda : \text{Fil}^\Lambda(\mathcal{X}) \rightarrow \text{Fil}^{\mathcal{I}_\Lambda}, \quad X^* \mapsto \{\underline{\text{Map}}_{\mathcal{X}}(\Sigma^\lambda B_i, X^{\lambda+2*})\}_{(i,\lambda) \in \mathcal{I}_\Lambda}.$$

We claim that $u_{\mathcal{B}_\Lambda}$ factors through w_Λ . As noted above, this claim reduces by Theorem 6.4.5 to factoring $\tau_{\geq 2*}^+ \circ \rho_{\mathcal{B}_\Lambda}$ through w_Λ . This latter factorization may be produced using the Λ -indexed Postnikov filtration in $\mathcal{X}$, defined as follows. For each $\lambda \in \Lambda$, we may transport the t-structure $(\mathcal{X}_{\geq 0}^{\mathcal{B}}, \mathcal{X}_{\leq 0}^{\mathcal{B}})$ along the automorphism Σ^λ to obtain another t-structure $(\mathcal{X}_{\geq \lambda}^{\mathcal{B}}, \mathcal{X}_{\leq \lambda}^{\mathcal{B}})$; let us write $\tau_{\geq \lambda}^{\mathcal{B}} : \mathcal{X} \rightarrow \mathcal{X}_{\geq \lambda}^{\mathcal{B}}$ for the corresponding connective cover functor, given by the formula

$$\tau_{\geq \lambda}^{\mathcal{B}}(X) := \Sigma^\lambda \tau_{\geq 0}^{\mathcal{B}}(\Sigma^{-\lambda} X).$$

The definition of the preorder on Λ may then be reformulated as follows: for $\lambda, \mu \in \Lambda$, we have $\mu \leq \lambda$ if and only if $\mathcal{X}_{\geq \lambda}^{\mathcal{B}} \subseteq \mathcal{X}_{\geq \mu}^{\mathcal{B}}$. It follows that the connective cover functors assemble into a functor $\tau_{\geq *}: \mathcal{X} \rightarrow \text{Fil}^\Lambda(\mathcal{X})$; in fact, this provides a factorization of $\tau_{\geq 2*} \circ \rho_{\mathcal{B}_\Lambda} : \mathcal{X}^{\mathcal{B}\text{-ev}} \rightarrow \text{Fil}^{\mathcal{I}_\Lambda}$ through w_Λ : for $E \in \mathcal{X}^{\mathcal{B}\text{-ev}}$, we have for each $(i, \lambda) \in \mathcal{I}_\Lambda$ a natural equivalence of filtered spectra

$$\begin{aligned} \underline{\text{Map}}_{\mathcal{X}}(\Sigma^\lambda B_i, \tau_{\geq \lambda+2*}(E)) &\simeq \underline{\text{Map}}_{\mathcal{X}}(\Sigma^\lambda B_i, \Sigma^\lambda \tau_{\geq 2*}(\Sigma^{-\lambda} E)) \\ &\simeq \underline{\text{Map}}_{\mathcal{X}}(B_i, \tau_{\geq 2*}(\Sigma^{-\lambda} E)) \\ &\simeq \tau_{\geq 2*} \underline{\text{Map}}_{\mathcal{X}}(B_i, \Sigma^{-\lambda} E) \\ &\simeq \tau_{\geq 2*} \underline{\text{Map}}_{\mathcal{X}}(\Sigma^\lambda B_i, E) \end{aligned}$$

(the third using projectivity of $\mathcal{B}$). Amalgamating this with constant filtrations in the same manner as in Construction 6.4.4 then defines the desired functor $\tau_{\geq 2*}^+ : \mathcal{X}^{\mathcal{B}\text{-ev}+} \rightarrow \text{Fil}^\Lambda(\mathcal{X})$ factoring $\tau_{\geq 2*}^+ \circ \rho_{\mathcal{B}_\Lambda}$ through w_Λ .

Finally, note that we may take $\mathcal{X} = \text{Spt}_G$ for a finite abelian group G and $\mathcal{B} = \{\Sigma_+^\infty G/H\}_{H \in \text{Sub}(G)}$, and take $\Lambda = \text{Rep}(G)$, acting on Spt_G through suspension by associated representation spheres, to recover the setup of Allen–Piessevaux reviewed in Example 6.2.3; in this case, the preceding discussion reproduces their G -equivariant perfect even filtration, valued in $\text{Rep}(G)$ -filtered G -spectra [1, §10].

Ch. A. Geometric stacks

This chapter reviews the aspects of the theory of (higher) stacks in the setting of classical algebraic geometry that are invoked in our account of the theory of derived commutative rings in §5.4.

§A.1. Stacks. We start with the basic definitions.

Notation A.1.1. We set $\mathbf{PStk} := \mathcal{P}^\vee(\mathbf{CRing})$, and we refer to objects of this ∞ -category, i.e. accessible functors $X : \mathbf{CRing} \rightarrow \mathbf{Spc}$, as *prestacks* (see Convention 0.3.10). We set $\mathbf{Spec} := \mathbf{j}_{\mathbf{CRing}}^\vee : \mathbf{CRing} \rightarrow \mathbf{PStk}^{\mathrm{op}}$ and refer to prestacks in the essential image of $\mathbf{Spec}$, i.e. the corepresentable functors, as *affine schemes*. We let $\mathbf{Stk} \subseteq \mathbf{PStk}$ denote the full subcategory spanned by those prestacks that are sheaves with respect to the fpqc topology (on $\mathbf{CRing}^{\mathrm{op}}$), which we refer to as *stacks*. The inclusion $\mathbf{Stk} \hookrightarrow \mathbf{PStk}$ admits a left adjoint, which we denote by $\mathbf{L}_{\mathrm{fpqc}} : \mathbf{PStk} \rightarrow \mathbf{Stk}$.

Notation A.1.2. For $A \in \mathbf{CRing}$, we let $\mathbf{Mod}_A$ denote the stable, presentable symmetric monoidal ∞ -category of A -module spectra. This carries a canonical t-structure, where an object is connective or coconnective if and only if its underlying spectrum is so; we denote the connective subcategory by $\mathbf{Mod}_A^{\mathrm{cn}}$. The t-structure is compatible with the symmetric monoidal structure, and the heart $\mathbf{Mod}_A^\vee$ identifies with the abelian symmetric monoidal category of ordinary A -modules. Moreover, $\mathbf{Mod}_A$ identifies with the derived ∞ -category $\mathbf{D}(\mathbf{Mod}_A^\vee)$.

Via base change of modules, these constructions extend to functors

$$\mathbf{Mod}, \mathbf{Mod}^{\mathrm{cn}}, \mathbf{Mod}^\vee : \mathbf{CRing} \rightarrow \mathbf{CAlg}(\mathbf{Pr}^{\mathrm{L}}),$$

together with natural inclusion and truncation transformations

$$\mathbf{Mod} \xleftarrow{\iota^{\mathrm{cn}}} \mathbf{Mod}^{\mathrm{cn}} \xrightarrow{\tau^\vee} \mathbf{Mod}^\vee.$$

Note that there are two distinct meanings of base change here: for a general map of commutative rings $A \rightarrow B$, the functor $B \otimes_A - : \mathbf{Mod}_A \rightarrow \mathbf{Mod}_B$ corresponds to the derived tensor product functor from $\mathbf{D}(\mathbf{Mod}_A^\vee)$ to $\mathbf{D}(\mathbf{Mod}_B^\vee)$, which preserves connectivity but need not carry $\mathbf{Mod}_A^\vee$ into $\mathbf{Mod}_B^\vee$; the classical base change functor $\mathbf{Mod}_A^\vee \rightarrow \mathbf{Mod}_B^\vee$ is obtained from this by truncation, as expressed by the naturality of $\tau^\vee$.

We let

$$\mathbf{QCoh}, \mathbf{QCoh}^{\mathrm{cn}}, \mathbf{QCoh}^\vee : \mathbf{PStk}^{\mathrm{op}} \rightarrow \mathbf{CAlg}(\mathbf{Pr}^{\mathrm{L}}), \quad \mathbf{QCoh} \xleftarrow{\iota^{\mathrm{cn}}} \mathbf{QCoh}^{\mathrm{cn}} \xrightarrow{\tau^\vee} \mathbf{QCoh}^\vee$$

denote the right Kan extension of the above functors and transformations along $\mathbf{Spec} : \mathbf{CRing} \rightarrow \mathbf{PStk}^{\mathrm{op}}$. For X a prestack, we refer to objects of $\mathbf{QCoh}(X)$ as *quasicoherent sheaves* on X , objects of $\mathbf{QCoh}^{\mathrm{cn}}(X)$ as *connective quasicoherent sheaves* on X and objects of $\mathbf{QCoh}^\vee(X)$ as *classical quasicoherent sheaves* on X .

Notation A.1.3. For $f : Y \rightarrow X$ a map of prestacks, we denote the image of f under $\mathbf{QCoh}$ by $f^* : \mathbf{QCoh}(X) \rightarrow \mathbf{QCoh}(Y)$. This is a map in $\mathbf{CAlg}(\mathbf{Pr}^{\mathrm{L}})$, i.e. a colimit preserving symmetric monoidal functor, and hence admits lax symmetric monoidal right adjoint, which we denote by $f_* : \mathbf{QCoh}(Y) \rightarrow \mathbf{QCoh}(X)$. Let us single out two special cases of these constructions, for which we will use special notation:

1. Given a prestack X and a commutative ring A , a point $x \in X(A)$ may be identified with a map $x : \mathbf{Spec}(A) \rightarrow X$, so for any quasicoherent sheaf $\mathcal{F} \in \mathbf{QCoh}(X)$, we obtain an A -module

$$\mathcal{F}(x) := x^*(\mathcal{F}) \in \mathbf{QCoh}(\mathbf{Spec}(A)) \simeq \mathbf{Mod}_A.$$

Roughly speaking, we may think of a quasicoherent sheaf $\mathcal{F}$ as the collection of these modules as A and x vary, together with data expressing their compatibility under base change. The picture is similar for discrete quasicoherent sheaves, although the warning in Notation A.1.2 concerning the meaning of base change should be kept in mind.

2. Suppose given a prestack Y and a map $p : Y \rightarrow \operatorname{Spec}(A)$ for a commutative ring A (e.g. the unique such map for $A = \mathbb{Z}$). For $\mathcal{F} \in \operatorname{QCoh}(Y)$, we may use the notation

$$\Gamma(Y; \mathcal{F}) := p_*(\mathcal{F}) \in \operatorname{QCoh}(\operatorname{Spec}(A)) \simeq \operatorname{Mod}_A,$$

and refer to this as the *global sections* of $\mathcal{F}$; this is given by the formula

$$\Gamma(Y; \mathcal{F}) \simeq \lim_{B \in \operatorname{CRing}, y \in Y(B)} \mathcal{F}(y).$$

As a lax symmetric monoidal functor, $\Gamma(Y; -)$ carries commutative algebra objects of $\operatorname{QCoh}(Y)$ to commutative algebra objects of Mod_A , i.e. objects of the ∞ -category CAlg_A of $\mathbb{E}_\infty$ - A -algebras.

Example A.1.4. Let X be a prestack. We denote the unit object of $\operatorname{QCoh}(X)$ by $\mathcal{O}_X$. For any commutative ring A and point $x \in X(A)$, we have $\mathcal{O}_X(x) \simeq A$. And we have

$$\Gamma(X; \mathcal{O}_X) \simeq \lim_{A \in \operatorname{CRing}, x \in X(A)} A;$$

this is by default an $\mathbb{E}_\infty$ - $\mathbb{Z}$ -algebra, and given a map of prestacks $p : X \rightarrow \operatorname{Spec}(R)$ for a commutative ring R , it obtains an $\mathbb{E}_\infty$ - R -algebra structure.

Remark A.1.5. Let X be a prestack. The functor $\iota^{\operatorname{cn}} : \operatorname{QCoh}^{\operatorname{cn}}(X) \rightarrow \operatorname{QCoh}(X)$ is fully faithful, and we will freely identify $\operatorname{QCoh}^{\operatorname{cn}}(X)$ with its image under this embedding. This defines the connective part of a t-structure on $\operatorname{QCoh}(X)$, with the coconnective subcategory $\operatorname{QCoh}(X)_{\leq 0} \subseteq \operatorname{QCoh}(X)$ formally consisting of those quasicoherent sheaves $\mathcal{G}$ such that $\operatorname{Map}_{\operatorname{QCoh}(X)}(\Sigma \mathcal{F}, \mathcal{G}) \simeq 0$ for every $\mathcal{F} \in \operatorname{QCoh}^{\operatorname{cn}}(X) = \operatorname{QCoh}(X)_{\geq 0}$.

By definition, a quasicoherent sheaf $\mathcal{F} \in \operatorname{QCoh}(X)$ is connective if and only if, for every commutative ring A and point $x \in X(A)$, the A -module $\mathcal{F}(x)$ is connective; and for any map $f : Y \rightarrow X$, the functor $f^* : \operatorname{QCoh}(X) \rightarrow \operatorname{QCoh}(Y)$ preserves connectivity. Beware that these comments do not apply in general for coconnectivity in place of connectivity, and relatedly, the heart of the above t-structure, $\operatorname{QCoh}(X)^\heartsuit$, does not a priori identify with the category $\operatorname{QCoh}^\heartsuit(X)$ of classical quasicoherent sheaves. We will see below that the situation is clearer under “geometricity” hypotheses on X .

Remark A.1.6. We make a couple of notes about the extent to which the above ∞ -categories of quasicoherent sheaves depend on the prestack X :

1. The theory of faithfully flat descent tells us that Mod , $\operatorname{Mod}^{\operatorname{cn}}$, and $\operatorname{Mod}^\heartsuit$ are fpqc sheaves. It follows that QCoh , $\operatorname{QCoh}^{\operatorname{cn}}$, and $\operatorname{QCoh}^\heartsuit$ factor through the localization $L_{\operatorname{fpqc}} : \operatorname{PStk} \rightarrow \operatorname{Stk}$.
2. Since $\operatorname{Mod}^\heartsuit$ takes values in the 2-category of presentable 1-categories, $\operatorname{QCoh}^\heartsuit$ furthermore factors through the 1-truncation functor $\tau_{\leq 1} : \operatorname{Stk} \rightarrow \operatorname{Stk}$ (which is given by applying the 1-truncation functor $\tau_{\leq 1} : \operatorname{Spc} \rightarrow \operatorname{Spc}$ pointwise and then applying L_{fpqc}).

§A.2. Geometricity. Now we narrow down our discussion to a more manageable class of stacks.

Definition A.2.1. Let $f : Y \rightarrow X$ be a map of stacks. We will inductively define for $n \in \mathbb{Z}_{\geq -1}$ what it means for f to be a *n-geometric* or *n-geometrically flat* or *n-geometrically faithfully flat*:

1. We say that f is *(-1)-geometric*, *(-1)-geometrically flat*, or *(-1)-geometrically faithfully flat* if it is an equivalence.
2. For $n \geq 0$, we say that f is a *n-geometric(ally (faithfully) flat)* if, for every commutative ring A and map $\operatorname{Spec}(A) \rightarrow X$, there exists a commutative ring B together with an *(n-1)-geometrically faithfully flat* map $\operatorname{Spec}(B) \rightarrow Y \times_X \operatorname{Spec}(A)$ (such that the induced map of commutative rings $A \rightarrow B$ is (faithfully) flat).

We say that f is a *geometric(ally (faithfully) flat)* if it is *n-geometric(ally (faithfully) flat)* for some $n \in \mathbb{Z}_{\geq -1}$.

Definition A.2.2. Let X be a stack. For $n \in \mathbb{Z}_{\geq 0}$, we say that X is *n-geometric* if the unique map $p : X \rightarrow \operatorname{Spec}(\mathbb{Z})$ is *n-geometric*. We say that X is *geometric* if it is *n-geometric* for some $n \in \mathbb{Z}_{\geq 0}$, i.e. if p is geometric.

Remark A.2.3. The above definitions go back to work of Simpson [37], and analogous definitions in the setting of derived algebraic geometry were formulated by Lurie [25] and Töen–Vezzosi [39]. Note though that we have implicitly included a quasicompactness hypothesis here, and that the indexing conventions vary. The analogue in spectral algebraic geometry of what we have called a “1-geometric stack” is also discussed at length by Lurie in [29, Chapter 9], but referred to there just by “geometric stack”.

We recall several basic properties of these notions in the following result.

Proposition A.2.4. *Let $n \in \mathbb{Z}_{\geq -1}$ and let $f : Y \rightarrow X$ be a map of stacks. Then the following statements hold.*

1. *Let $p : X' \rightarrow X$ be a map of stacks and let $f' : Y' \rightarrow X'$ be the base change of f along p . If f is n -geometric(ally (faithfully) flat), then f' is n -geometric(ally (faithfully) flat). Moreover, the converse also holds when p is fpqc-locally surjective.*
2. *Suppose that f is n -geometrically flat. Then f is n -geometrically faithfully flat if and only if it is fpqc-locally surjective.*
3. *Suppose that $n \geq 0$ and that f is n -geometrically (faithfully) flat. Suppose given commutative rings A and B , a map $\text{Spec}(A) \rightarrow X$, and any $(n-1)$ -geometrically faithfully flat map $y : \text{Spec}(B) \rightarrow Y \times_X \text{Spec}(A)$. Then the induced map of commutative rings $A \rightarrow B$ is (faithfully) flat.*
4. *Suppose that f is n -geometric and let $m \geq n$. Then f is m -geometric. Furthermore, f is n -geometrically (faithfully) flat if and only if it is m -geometrically (faithfully) flat.*

For the remaining statements, we let $g : Z \rightarrow Y$ be another map of stacks and set $h := fg : Z \rightarrow X$.

5. *Suppose that f and g are n -geometric(ally (faithfully) flat). Then h is n -geometric(ally (faithfully) flat).*
6. *Suppose that $n \geq 0$, that g is $(n-1)$ -geometrically faithfully flat, and that h is n -geometric(ally (faithfully) flat). Then f is n -geometric(ally (faithfully) flat).*
7. *Suppose that $n \geq 0$, that f is 0-geometric, and that h is n -geometric. Then g is n -geometric.*
8. *Suppose that $n \geq 1$, that f is n -geometric, and that h is $(n-1)$ -geometric. Then g is $(n-1)$ -geometric.*

Proof. The first claim of 1 is clear from the definitions. We will return to the converse later in the proof.

We now prove 2, by induction on n . It follows from the definitions that f is n -geometrically faithfully flat or fpqc-locally surjective if and only if the base change $Y \times_X \text{Spec}(A) \rightarrow \text{Spec}(A)$ is so for any commutative ring A and map $\text{Spec}(A) \rightarrow X$, so we may assume that $X = \text{Spec}(A)$. The case $n = -1$ is tautological, so suppose $n \geq 0$. We may choose an $(n-1)$ -geometrically faithfully flat map $y : \text{Spec}(B) \rightarrow Y$ such that the induced map $A \rightarrow B$ is flat. By the inductive hypothesis, y is fpqc-locally surjective. Thus f is fpqc-locally surjective if and only if $fy : \text{Spec}(B) \rightarrow \text{Spec}(A)$ is fpqc-locally surjective. This reduces us to the claim that the flat map of commutative rings $A \rightarrow B$ is faithfully flat if and only if the induced map $\text{Spec}(B) \rightarrow \text{Spec}(A)$ is fpqc-locally surjective. The “only if” direction is clear, and the “if” direction follows from the fact that pullback of quasicoherent sheaves along an fpqc-locally surjective map is conservative.

We next prove 3, again inducting on n . We may replace X by $\text{Spec}(A)$; let us do so to simplify notation. By hypothesis, we may find a commutative ring B' and an $(n-1)$ -geometrically faithfully flat map $y' : \text{Spec}(B') \rightarrow Y$ such that the induced map $A \rightarrow B'$ is (faithfully) flat. In the base case $n = 0$, the maps y and y' are equivalences, so the induced map $A \rightarrow B$ must also be (faithfully) flat. Suppose then that $n \geq 1$. Form a cartesian diagram

$$\begin{array}{ccc} W & \xrightarrow{r} & \text{Spec}(B) \\ r' \downarrow & & \downarrow y \\ \text{Spec}(B') & \xrightarrow{y'} & Y. \end{array}$$

By the direction of 1 that has been proven, r and r' are $(n-1)$ -geometrically faithfully flat. Thus, we may choose an $(n-2)$ -geometrically faithfully flat map $\text{Spec}(C) \rightarrow W$, and by the inductive hypothesis, both induced maps $B \rightarrow C$ and $B' \rightarrow C$ are faithfully flat. From the latter we deduce that the composite map $A \rightarrow C$ is (faithfully) flat, and then using the former we deduce that $A \rightarrow B$ is (faithfully) flat.

We now begin proving 4: we will prove the first claim and the “only if” direction of the second claim

now, and return to the “if” direction of the second claim at the end of the proof. Induction on m reduces us to the case $m = n + 1$, and we may again assume that $X = \operatorname{Spec}(A)$ for a commutative ring A . Additionally, in view of 2, the statement with faithful flatness follows immediately from the the statement with flatness. We now proceed by induction on n . The base case $n = -1$ is clear, so suppose that $n \geq 0$. Then we may choose a commutative ring B and an $(n - 1)$ -geometrically faithfully flat map $y : \operatorname{Spec}(B) \rightarrow Y$. By the inductive hypothesis, y is n -geometrically faithfully flat, showing that f is $(n + 1)$ -geometric. We similarly see that if f is n -geometrically flat then it is $(n + 1)$ -geometrically flat.

We now prove 5, by induction on n . The base case $n = -1$ is clear. Suppose that $n \geq 0$, and suppose given a commutative ring A and a map $x : \operatorname{Spec}(A) \rightarrow X$. Then we may form a commutative diagram of stacks,

$$\begin{array}{ccccccc}
 \operatorname{Spec}(C) & \xrightarrow{z} & Z'' & \xrightarrow{y'} & Z' & \xrightarrow{x''} & Z \\
 & \searrow & \downarrow g'' & & \downarrow g' & & \downarrow g \\
 & & \operatorname{Spec}(B) & \xrightarrow{y} & Y' & \xrightarrow{x'} & Y \\
 & & & \searrow & \downarrow f' & & \downarrow f \\
 & & & & \operatorname{Spec}(A) & \xrightarrow{x} & X,
 \end{array}$$

where each square is cartesian, and where we have chosen a commutative ring B and $(n - 1)$ -geometrically faithfully flat map y (such that the induced map $A \rightarrow B$ is (faithfully) flat) and a commutative ring C and $(n - 1)$ -geometrically faithfully flat map z (such that the induced map $B \rightarrow C$ is (faithfully) flat). Then the base change y' of y is also $(n - 1)$ -geometrically faithfully flat. By the inductive hypothesis, the composition $y'z$ is $(n - 1)$ -geometrically faithfully flat (and the induced map $A \rightarrow C$ is (faithfully) flat), proving the claim.

Let us now prove the converse statement from 1, so we assume that p is fpqc-locally surjective and that f' is n -geometric(ally (faithfully) flat). There will be two cases, but we will not need to induct. The case $n = -1$ follows from the hypothesis that X and Y are stacks, and hence may be recovered as the colimits in Stk of the Čech nerves of p and its base change $q : Y' \rightarrow Y$. Suppose then that $n \geq 0$. We may assume that $X = \operatorname{Spec}(A)$ and $X' = \operatorname{Spec}(A')$ for commutative rings A and A' , with p corresponding to a faithfully flat map $A \rightarrow A'$. Then we may choose a commutative ring B and an $(n - 1)$ -geometrically faithfully flat map $y' : \operatorname{Spec}(B) \rightarrow Y'$ (such that the induced map $A' \rightarrow B$ is (faithfully) flat). The base change $q : Y' \rightarrow Y$ is 0-geometrically faithfully flat, and hence $(n - 1)$ -geometrically faithfully flat by the direction of 4 proved above. Applying 5, we deduce that the composition $qy' : \operatorname{Spec}(B) \rightarrow Y$ is $(n - 1)$ -geometrically faithfully flat (with the induced map $A \rightarrow B$ being (faithfully) flat), as desired.

We move on to 6. We will induct on n , the base case $n = 0$ being clear. Suppose that $n \geq 1$. We may assume that $X = \operatorname{Spec}(A)$ for a commutative ring A . Since h is n -geometric(ally (faithfully) flat), we may choose a commutative ring B together with an $(n - 1)$ -geometrically faithfully flat map $z : \operatorname{Spec}(B) \rightarrow Z$ (such that the induced map $A \rightarrow B$ is (faithfully) flat). By 5, the composition $gz : \operatorname{Spec}(B) \rightarrow Y$ is also $(n - 1)$ -geometrically faithfully flat, which finishes.

Next we prove 7. We may assume that $X = \operatorname{Spec}(A)$ for a commutative ring A , and then f being 0-geometric implies that $Y = \operatorname{Spec}(B)$ for a commutative ring B , and then h being n -geometric implies that we may find a commutative ring C and an $(n - 1)$ -geometrically faithfully flat map $z : \operatorname{Spec}(C) \rightarrow Z$. This also shows that g is n -geometric.

Now we prove 8. We may reduce to the situation that $X = \operatorname{Spec}(A)$ for a commutative ring A . Since h is $(n - 1)$ -geometric, we may choose a commutative ring B and an $(n - 2)$ -geometrically faithfully flat map $z : \operatorname{Spec}(B) \rightarrow Z$. By 6, it suffices to show that the composite map $y := gz : \operatorname{Spec}(B) \rightarrow Y$ is $(n - 1)$ -geometric. Since f is n -geometric, we may choose a commutative ring B' and an $(n - 1)$ -geometrically

faithfully flat map $y' : \mathrm{Spec}(B') \rightarrow Y$. Form a cartesian diagram

$$\begin{array}{ccc} W & \xrightarrow{r} & \mathrm{Spec}(B) \\ r' \downarrow & & \downarrow y \\ \mathrm{Spec}(B') & \xrightarrow{y'} & Y. \end{array}$$

By 1, r is $(n-1)$ -geometrically faithfully flat, in particular $(n-1)$ -geometric. Since $\mathrm{Spec}(B)$ is 0-geometric, and hence $(n-1)$ -geometric by the part of statement 4 that has been proven, it follows from 5 that W is $(n-1)$ -geometric. Since $\mathrm{Spec}(B')$ is also 0-geometric, we can then deduce from 7 that r' is $(n-1)$ -geometric. By 2, y' is an fpqc-local surjection, so by the converse direction of 1 we conclude that y is $(n-1)$ -geometric, as desired.

It remains only to prove the “if” direction of the second claim in 4. The reductions noted in the earlier discussion of this statement apply here too: we may take $m = n + 1$ and $X = \mathrm{Spec}(A)$ for a commutative ring A , and it suffices to prove the assertion for flatness. The case $n = -1$ is tautological, so assume that $n \geq 0$ and that f is $(n+1)$ -geometrically flat. The latter means that we may choose a commutative ring C and an n -geometrically faithfully flat map $y : \mathrm{Spec}(C) \rightarrow Y$ such that the induced map $A \rightarrow C$ is flat. If $n = 0$, then f being 0-geometric means that $Y \simeq \mathrm{Spec}(B)$ for some commutative ring B , and y being 0-geometrically faithfully flat means that the induced map $B \rightarrow C$ is faithfully flat. The map $A \rightarrow B$ must then be flat, proving that f is 0-geometrically flat.

Finally, we handle the case $n \geq 1$ by induction. The composite map $\mathrm{Spec}(C) \rightarrow \mathrm{Spec}(A)$ is 0-geometric, hence $(n-1)$ -geometric, so 8 implies that y is $(n-1)$ -geometric. Now the inductive hypothesis guarantees that y is $(n-1)$ -geometrically faithfully flat, and hence exhibits f as n -geometrically flat. $\square$

Corollary A.2.5. *Let $n \in \mathbb{Z}_{\geq 0}$ and let X be a stack. Suppose given n -geometric maps $f_i : Y_i \rightarrow X$ for $i \in \{0, 1, 2\}$ and maps $g_1 : Y_1 \rightarrow Y_0$ and $g_2 : Y_2 \rightarrow Y_0$ over X . Let $f : Y_1 \times_{Y_0} Y_2 \rightarrow X$ be the induced map. Then:*

1. f is n -geometric;
2. if f_1 and g_2 are n -geometrically (faithfully) flat, then f is n -geometrically (faithfully) flat.

Proof. By Proposition A.2.4.8, g_1 and g_2 are n -geometric. By Proposition A.2.4.1, the base change $g'_2 : Y_1 \times_{Y_0} Y_2 \rightarrow Y_1$ of g_2 along g_1 is n -geometric. And finally by Proposition A.2.4.5, the composition $f_1 \circ g'_2 \simeq f$ is n -geometric. This proves 1, and similar reasoning proves 2. $\square$

Corollary A.2.6. *Let $n \in \mathbb{Z}_{\geq 0}$ and let X be a stack. Let $f_\bullet : Y_\bullet \rightarrow X$ be a groupoid object in Stk/X and let $f : Y \rightarrow X$ be its colimit. Suppose that f_0 is n -geometric(ally (faithfully) flat) and that the face map $d_0 : Y_1 \rightarrow Y_0$ is $(n-1)$ -geometrically faithfully flat. Then:*

1. the map f is n -geometric(ally (faithfully) flat);
2. for each nonnegative integer m , the map f_m is $(n-1)$ -geometric(ally (faithfully) flat);
3. for every injective map $[m] \hookrightarrow [m']$ in Δ , the induced map $Y_{m'} \rightarrow Y_m$ is $(n-1)$ -geometrically faithfully flat.

Proof. We begin with 1. Let $p : Y_0 \rightarrow Y$ be the augmentation, so that $f \circ p \simeq f_0$. Since f_0 is $(n-1)$ -geometric(ally (faithfully) flat), it is n -geometric(ally (faithfully) flat) by Proposition A.2.4.4. Thus, to prove that f is n -geometric(ally (faithfully) flat), it suffices by Proposition A.2.4.6 to prove that p is $(n-1)$ -geometrically faithfully flat. Since p is fpqc-locally surjective, it furthermore suffices by Proposition A.2.4.1 to show that the base change of p along itself is $(n-1)$ -geometrically faithfully flat. Since $U_\bullet$ is a groupoid object, this base change identifies with the face map d_0 , which is $(n-1)$ -geometrically faithfully flat by hypothesis.

The assertion in 2 is true by hypothesis for $m = 0$. From this it follows for $m = 1$ using Proposition A.2.4.5 and the fact that $d_0 : Y_1 \rightarrow Y_0$ is $(n-1)$ -geometric(ally (faithfully) flat). Then it follows for $m \geq 2$ by Corollary A.2.5, since $f_\bullet$ is a groupoid object.

Finally, for 3, it suffices by Proposition A.2.4.5 to treat the case of a face map $Y_{m+1} \rightarrow Y_m$. Since $Y_\bullet$ is

a groupoid object, each of these is a base change of d_0 , and hence is $(n-1)$ -geometrically faithfully flat by Proposition A.2.4.1. $\square$

Example A.2.7. Let R be a commutative ring and let A be a flat commutative R -algebra. Equipping $\mathrm{Spec}(A)$ with a group structure in $\mathrm{Stk}/_{\mathrm{Spec}(R)}$ is equivalent to promoting A to a commutative Hopf algebra over R . Given this structure, we call $G := \mathrm{Spec}(A)$ a *flat affine group scheme over R* . We may then form the bar construction/classifying stack $BG \simeq \mathrm{colim}_{[m] \in \Delta_{\mathrm{op}}} G^{\times m} \in \mathrm{Stk}/_{\mathrm{Spec}(R)}$, and by Corollary A.2.6, this is a 1-geometric stack, in fact 1-geometrically flat over $\mathrm{Spec}(R)$. If the group structure on G is commutative, i.e. A is a bicommutative Hopf algebra over R , then this construction may be iterated: we find that $B^n G$ is n -geometrically flat over $\mathrm{Spec}(R)$ for each nonnegative integer n .

Remark A.2.8. Let $n \in \mathbb{Z}_{\geq 0}$ and let Y be an n -geometric stack. That means that we may choose a commutative ring A and an $(n-1)$ -geometrically faithfully flat map $y : \mathrm{Spec}(A) \rightarrow Y$. Let $Y_\bullet$ be the Čech nerve of y . By Proposition A.2.4.2, y is fpqc-locally surjective, and so Y is the geometric realization (in Stk) of $Y_\bullet$. We have $Y_0 \simeq \mathrm{Spec}(A)$, which is 0-geometric and hence $(n-1)$ -geometric by Proposition A.2.4.4. And the face map $d_0 : Y_1 \rightarrow Y_0$ is a base change of y , and hence $(n-1)$ -geometrically faithfully flat by Proposition A.2.4.1. Thus, this gives a presentation of Y of the type in Corollary A.2.6 (where here we have $X = \mathrm{Spec}(\mathbb{Z})$).

We now review some implications of geometricity on the behavior of quasicoherent sheaves (adapting results from [29, §9.1]).

Proposition A.2.9. *Let X be a geometric stack. Then the following statements hold:*

1. *The t -structure on $\mathrm{QCoh}(X)$ is right complete and compatible with filtered colimits (equivalently, the canonical functor $\mathrm{QCoh}(X) \rightarrow \mathrm{Spt}(\mathrm{QCoh}^{\mathrm{cn}}(X))$ is an equivalence and $\mathrm{QCoh}^{\mathrm{cn}}(X)$ is a Grothendieck prestack ∞ -category [29, Remark C.3.1.5]).*
2. *The truncation functor $\tau^\heartsuit : \mathrm{QCoh}^{\mathrm{cn}}(X) \rightarrow \mathrm{QCoh}^\heartsuit(X)$ admits a fully faithful right adjoint, with essential image $\mathrm{QCoh}(X)^\heartsuit$.*
3. *For any geometrically flat map $f : Y \rightarrow X$, the pullback functor $f^* : \mathrm{QCoh}(X) \rightarrow \mathrm{QCoh}(Y)$ is t -exact (equivalently, the functor $f^* : \mathrm{QCoh}^{\mathrm{cn}}(X) \rightarrow \mathrm{QCoh}^{\mathrm{cn}}(Y)$ is left exact [29, Proposition C.3.2.1]).*

Proof. By Proposition A.2.4.4, it suffices to show for any $n \in \mathbb{Z}_{\geq 0}$ that the claims hold when X is an n -geometric stack and f is an n -geometrically flat map. We will proceed by induction on n . The claims are clear in the base case $n = 0$, where $X \simeq \mathrm{Spec}(A)$ and $Y \simeq \mathrm{Spec}(B)$ for commutative rings A and B and the map $A \rightarrow B$ corresponding to f is flat.

Suppose now that $n \geq 1$. We choose a commutative ring A and an $(n-1)$ -geometrically faithfully flat map $x : \mathrm{Spec}(A) \rightarrow X$ and let $X_\bullet$ be the Čech nerve of x . By Remark A.2.8, for each $[m] \in \Delta$, the stack X_m is $(n-1)$ -geometric, and for each injective map $[m] \hookrightarrow [m']$ in Δ , the associated map $X_{m'} \rightarrow X_m$ is $(n-1)$ -geometrically faithfully flat. Thus, by the inductive hypothesis, 1 and 2 hold when X is replaced by any of the stacks X_m , and 3 holds when f is replaced by any of the maps $X_{m'} \rightarrow X_m$ as above.

Now, since x is fpqc-locally surjective (Proposition A.2.4.2), we have equivalences

$$\begin{aligned} \mathrm{QCoh}(X) &\xrightarrow{\sim} \lim_{[m] \in \Delta_s} \mathrm{QCoh}(X_m), \\ \mathrm{QCoh}^{\mathrm{cn}}(X) &\xrightarrow{\sim} \lim_{[m] \in \Delta_s} \mathrm{QCoh}^{\mathrm{cn}}(X_m), \\ \mathrm{QCoh}^\heartsuit(X) &\xrightarrow{\sim} \lim_{[m] \in \Delta_s} \mathrm{QCoh}^\heartsuit(X_m). \end{aligned}$$

Using that the pullback functor $\mathrm{QCoh}^{\mathrm{cn}}(X_m) \rightarrow \mathrm{QCoh}^{\mathrm{cn}}(X_{m'})$ induced by any map $[m] \hookrightarrow [m']$ in Δ_s is left exact, the first two equivalences together with [29, Proposition C.3.2.4] prove 1, and additionally that $x^* : \mathrm{QCoh}(X) \rightarrow \mathrm{Mod}_A \simeq \mathrm{QCoh}(X_0)$ is t -exact. Combining the left exactness used in the previous

sentence with the fact that 2 holds for each X_m , we deduce that the commutative square

$$\begin{array}{ccc} \mathrm{QCoh}^{\mathrm{cn}}(X_m) & \xrightarrow{\tau^\vee} & \mathrm{QCoh}^\vee(X_m) \\ \downarrow & & \downarrow \\ \mathrm{QCoh}^{\mathrm{cn}}(X_{m'}) & \xrightarrow{\tau^\vee} & \mathrm{QCoh}^\vee(X_{m'}) \end{array}$$

induced by any map $[m] \mapsto [m']$ in Δ_s is right adjointable, in the sense of [26, Definition 4.7.4.13]. Applying [26, Corollary 4.7.4.18] (or Proposition 1.1.10), this implies that the right adjoint to $\tau^\vee : \mathrm{QCoh}^{\mathrm{cn}}(X) \rightarrow \mathrm{QCoh}^\vee(X)$ identifies with the limit over $[m] \in \Delta_s$ of the inclusions $\mathrm{QCoh}(U_m)^\vee \hookrightarrow \mathrm{QCoh}^{\mathrm{cn}}(U_m)$, and hence is fully faithful with essential image given by those objects $\mathcal{F} \in \mathrm{QCoh}(X)$ such that $x^*(\mathcal{F}) \in \mathrm{Mod}_A^\vee$. Since x^* is t-exact and conservative, this last condition holds if and only if $\mathcal{F} \in \mathrm{QCoh}(X)^\vee$, proving 2.

We now prove 3. Since X and f are n -geometric, so is Y (Proposition A.2.4.5). Choosing a commutative ring B and an $(n-1)$ -geometrically faithfully flat map $y : \mathrm{Spec}(B) \rightarrow Y$, the argument above shows that y^* is t-exact and conservative. We may thus replace Y by $\mathrm{Spec}(B)$. Now form a cartesian diagram

$$\begin{array}{ccc} Z & \xrightarrow{f'} & \mathrm{Spec}(A) \\ x' \downarrow & & \downarrow x \\ \mathrm{Spec}(B) & \xrightarrow{f} & X. \end{array}$$

Since x is $(n-1)$ -geometrically faithfully flat, so is x' (Proposition A.2.4.1); in particular, Z is $(n-1)$ -geometric (Proposition A.2.4.5). By Proposition A.2.4.8 and Proposition A.2.4.4, f is $(n-1)$ -geometrically flat, and so f' is too (Proposition A.2.4.1 again). By the inductive hypothesis, $(x')^*$ and $(f')^*$ are t-exact, and we also showed above that x^* is t-exact. Since $(x')^*$ is also conservative, we deduce that f^* is t-exact, as desired. $\square$

Proposition A.2.10. *Let $p : U \rightarrow X$ be a 0-geometric map of stacks. Then the following statements hold:*

1. *The functor $p_* : \mathrm{QCoh}(U) \rightarrow \mathrm{QCoh}(X)$ preserves colimits.*
2. *For $\mathcal{F} \in \mathrm{QCoh}(X)$ and $\mathcal{G} \in \mathrm{QCoh}(Y)$, the canonical projection map $\mathcal{F} \otimes p_*(\mathcal{G}) \rightarrow p_*(p^*(\mathcal{F}) \otimes \mathcal{G})$ is an equivalence.*
3. *Suppose given a cartesian diagram*

$$\begin{array}{ccc} V & \xrightarrow{q} & Y \\ g \downarrow & & \downarrow f \\ U & \xrightarrow{p} & X. \end{array}$$

*Then, for any $\mathcal{G} \in \mathrm{QCoh}(U)$, the canonical base change map $f^*p_*(\mathcal{G}) \rightarrow q_*g^*(\mathcal{G})$ is an equivalence.*

Proof. Suppose first that $X \simeq \mathrm{Spec}(A)$ and $Y \simeq \mathrm{Spec}(B)$ for commutative rings A and B . Since p is 0-geometric, our cartesian diagram then corresponds to a pushout diagram of commutative rings

$$\begin{array}{ccc} D & \longleftarrow & B \\ \uparrow & & \uparrow \\ C & \longleftarrow & A. \end{array}$$

The functor p_* corresponds to the forgetful functor $\mathrm{Mod}_C \rightarrow \mathrm{Mod}_A$, which preserves colimits, proving 1 in this case. For $\mathcal{F} \in \mathrm{QCoh}(X) \simeq \mathrm{Mod}_A$ and $\mathcal{G} \in \mathrm{QCoh}(U) \simeq \mathrm{Mod}_C$, the projection map $\mathcal{F} \otimes p_*(\mathcal{G}) \rightarrow p_*(p^*(\mathcal{F}) \otimes \mathcal{G})$ identifies with the equivalence $\mathcal{F} \otimes_A \mathcal{G} \simeq (\mathcal{F} \otimes_A C) \otimes_C \mathcal{G}$, and the base change map $f^*p_*(\mathcal{G}) \rightarrow q_*g^*(\mathcal{G})$ identifies with the equivalence $\mathcal{G} \otimes_A B \xrightarrow{\sim} \mathcal{G} \otimes_C D$, proving 2 and 3 in this case.

We next prove 3 for general X but still with $Y \simeq \mathrm{Spec}(B)$. In fact, this follows from the special case of the previous paragraph and [26, Corollary 4.7.4.18] (or Proposition 1.1.10), since p is the colimit over all maps f from such Y to X of the base change q , and hence p^* is the limit over all such f of q^* .

Now we prove 3 for general X and general Y . Let $y : Y' \rightarrow Y$ be any map with $Y' \simeq \operatorname{Spec}(B)$ for a commutative ring B , and form the further pullback

$$\begin{array}{ccc} V' & \xrightarrow{q'} & Y' \\ v \downarrow & & \downarrow y \\ V & \xrightarrow{q} & Y \\ g \downarrow & & \downarrow f \\ U & \xrightarrow{p} & X; \end{array}$$

let $f' := fy : Y' \rightarrow X$ and $g' := gv : V' \rightarrow U$. Then, for any $\mathcal{G} \in \operatorname{QCoh}(U)$, we have a commutative diagram

$$\begin{array}{ccc} y^* f^* p_*(\mathcal{G}) & \xrightarrow{\quad} & y^* q_* g^*(\mathcal{G}) \\ \sim \downarrow & & \downarrow \\ (f')^* p_*(\mathcal{G}) & \longrightarrow & q'_*(g')^*(\mathcal{G}) \xrightarrow{\sim} q'_* v^* g^*(\mathcal{G}) \end{array}$$

By the previous paragraph, the lower left horizontal map and right vertical map are equivalences, and hence the upper horizontal map is an equivalence. This proves the claim, since $\operatorname{QCoh}(Y)$ is the limit over all such maps y of $\operatorname{QCoh}(Y')$.

Finally, let us prove 1 and 2 for general X . Similar to the previous argument, it suffices to show that, for any cartesian diagram as in 3 where $Y \simeq \operatorname{Spec}(B)$ for a commutative ring B , the composite functor $f^* p_* : \operatorname{QCoh}(U) \rightarrow \operatorname{QCoh}(Y)$ preserves colimits and f^* carries the projection map to an equivalence. For the former, having proven 3, $f^* p_*$ identifies with the composition

$$\operatorname{QCoh}(U) \xrightarrow{g^*} \operatorname{QCoh}(V) \xrightarrow{q_*} \operatorname{QCoh}(Y);$$

we know that g^* preserves colimits, and q_* preserves colimits by the case of 1 established in the first paragraph of the proof. For the projection map, we have a commutative diagram

$$\begin{array}{ccc} f^*(\mathcal{F} \otimes p_*(\mathcal{G})) & \xrightarrow{\quad} & f^* p_*(p^*(\mathcal{F}) \otimes \mathcal{G}) \\ \downarrow \sim & & \searrow \\ f^*(\mathcal{F}) \otimes f^* p_*(\mathcal{G}) & & q_* g^*(p^*(\mathcal{F}) \otimes \mathcal{G}) \\ \downarrow & & \downarrow \sim \\ f^*(\mathcal{F}) \otimes q_* g^*(\mathcal{G}) & \longrightarrow & q_*(q^* f^*(\mathcal{F}) \otimes g^*(\mathcal{G})) \xrightarrow{\sim} q_*(g^* p^*(\mathcal{F}) \otimes g^*(\mathcal{G})); \end{array}$$

the lower left vertical and upper right diagonal maps are equivalences by 3, and the lower left horizontal map is an equivalence by the case of 2 established in the first paragraph of the proof, so the upper horizontal map is an equivalence too, as desired. $\square$

Lemma A.2.11. *Let X be a geometric stack. Then the functor $\Gamma(X; -) : \operatorname{QCoh}(X)_{\leq 0} \rightarrow (\operatorname{Mod}_{\mathbb{Z}})_{\leq 0}$ preserves filtered colimits.*

Proof. Let $n \in \mathbb{Z}_{\geq 0}$ be such that X is an n -geometric stack. We will prove the claim by induction on n . When $n = 0$, we have an equivalence $X \simeq \operatorname{Spec}(A)$, and the functor $\Gamma(X; -) : \operatorname{QCoh}(X) \rightarrow \operatorname{Mod}_{\mathbb{Z}}$ identifies with the forgetful functor $\operatorname{Mod}_A \rightarrow \operatorname{Mod}_{\mathbb{Z}}$, which preserves colimits, whence the restriction to $\operatorname{QCoh}(X)_{\leq 0} \simeq (\operatorname{Mod}_A)_{\leq 0}$ preserves filtered colimits.

Assume now that $n \geq 1$. By Remark A.2.8, we may write X as the geometric realization of a simplicial stack $X_\bullet$ in which each object X_m is $(n-1)$ -geometric and each face map is geometrically flat. For each nonnegative integer m , let $p_m : X_m \rightarrow X$ denote the associated projection map. It follows from Proposition A.2.9.3 that the pullback functors $p_m^* : \operatorname{QCoh}(X) \rightarrow \operatorname{QCoh}(X_m)$ are t-exact. They also induce

an equivalence $\mathrm{QCoh}(X) \simeq \lim_{[m] \in \Delta} \mathrm{QCoh}(X_m)$, and hence a natural equivalence

$$\Gamma(X; \mathcal{F}) \simeq \lim_{[m] \in \Delta} \Gamma(X_m, p_m^*(\mathcal{F}))$$

for $\mathcal{F} \in \mathrm{QCoh}(X)$. By the inductive hypothesis, $\Gamma(X_m; -) : \mathrm{QCoh}(X_m)_{\leq 0} \rightarrow (\mathrm{Mod}_{\mathbb{Z}})_{\leq 0}$ preserves filtered colimits for each nonnegative integer m . Since totalizations commute with filtered colimits in $(\mathrm{Mod}_{\mathbb{Z}})_{\leq 0}$, we deduce that $\Gamma(X; -) : \mathrm{QCoh}(X)_{\leq 0} \rightarrow (\mathrm{Mod}_{\mathbb{Z}})_{\leq 0}$ preserves filtered colimits too. $\square$

Proposition A.2.12. *Suppose given geometric stacks X and Y and a map $f : Y \rightarrow X$. Then the following statements hold:*

1. *The functor $f_* : \mathrm{QCoh}(Y)_{\leq 0} \rightarrow \mathrm{QCoh}(X)_{\leq 0}$ preserves filtered colimits.*
2. *Suppose given a cartesian diagram*

$$\begin{array}{ccc} V & \xrightarrow{q} & Y \\ g \downarrow & & \downarrow f \\ U & \xrightarrow{p} & X \end{array}$$

in which the map p is geometrically flat, and let $\mathcal{F} \in \mathrm{QCoh}(Y)_{\leq 0}$. Then the canonical map $p^ f_*(\mathcal{F}) \rightarrow g_* q^*(\mathcal{F})$ is an equivalence.*

Proof. By Proposition A.2.4.4, it suffices to show for any $n \geq 0$ that the claims hold when X is n -geometric and p is n -geometrically flat. We will prove this by induction on n .

Suppose that $n = 0$, so that $X \simeq \mathrm{Spec}(A)$ and $U \simeq \mathrm{Spec}(C)$ for commutative rings A and C , with the map $A \rightarrow C$ corresponding to p being flat. Let us first prove 1. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{QCoh}(Y) & \xrightarrow{f_*} & \mathrm{QCoh}(X) \\ & \searrow \Gamma(Y; -) & \swarrow \Gamma(X; -) \\ & \mathrm{Mod}_{\mathbb{Z}} & \end{array}$$

The functor $\Gamma(X; -)$ identifies with the forgetful functor $\mathrm{Mod}_A \rightarrow \mathrm{Mod}_{\mathbb{Z}}$, which is conservative and colimit preserving. Thus, to show that $f_* : \mathrm{QCoh}(Y)_{\leq 0} \rightarrow \mathrm{QCoh}(X)_{\leq 0}$ preserves colimits, it suffices to show that $\Gamma(Y; -) : \mathrm{QCoh}(Y)_{\leq 0} \rightarrow (\mathrm{Mod}_{\mathbb{Z}})_{\leq 0}$ preserves colimits. This is true by Lemma A.2.11.

We now prove 2. The functor $p_* : \mathrm{QCoh}(U) \rightarrow \mathrm{QCoh}(X)$ is the forgetful functor $\mathrm{Mod}_C \rightarrow \mathrm{Mod}_A$, which is conservative, so it suffices to show that the induced map $\phi : p_* p^* f_*(\mathcal{F}) \rightarrow p_* g_* q^*(\mathcal{F})$ is an equivalence. We have equivalences

$$p_* g_* q^*(\mathcal{F}) \simeq f_* q_* q^*(\mathcal{F}) \simeq f_*(q_*(\mathcal{O}_V) \otimes \mathcal{F}) \simeq f_*(f^*(C) \otimes \mathcal{F}),$$

the second and third coming from Proposition A.2.10. Given these, ϕ identifies with the projection map $C \otimes f_*(\mathcal{F}) \rightarrow f_*(f^*(C) \otimes \mathcal{F})$. Consider the collection of A -modules M for which the projection map $M \otimes f_*(\mathcal{F}) \rightarrow f_*(f^*(M) \otimes \mathcal{F})$ is an equivalence. It contains A and is closed under finite sums, and by 1, it is closed under filtered colimits. Thus, by Lazard's theorem, it contains all flat A -modules, in particular C .

Now suppose that $n \geq 1$. This time we will first prove 2. Furthermore, let us begin with the special case where $U \simeq \mathrm{Spec}(C)$ for some commutative ring C . Note that then p is necessarily $(n-1)$ -geometrically flat, by Proposition A.2.4.8. By Remark A.2.8, we may write X as the geometric realization of a simplicial stack $X_\bullet$ in which each object X_m is $(n-1)$ -geometric and each face map is geometrically flat, and the projection map $x_0 : X_0 \rightarrow X$ is $(n-1)$ -geometrically faithfully flat. We then form a commutative cube

$$\begin{array}{ccccc} & & V_0 & \xrightarrow{q_0} & Y_0 \\ & \swarrow v_0 & \downarrow q & \swarrow y_0 & \downarrow f_0 \\ V & & & & Y \\ & \searrow g_0 & \downarrow p_0 & \searrow f & \downarrow x_0 \\ & & U_0 & \xrightarrow{p_0} & X_0 \\ & \swarrow u_0 & \downarrow p & \swarrow x_0 & \\ U & & & & X \end{array}$$

where each face is cartesian. By Proposition A.2.4.1, u_0 is $(n-1)$ -geometrically faithfully flat and p_0 is $(n-1)$ -geometrically flat. Now consider the induced commutative diagram

$$\begin{array}{ccc} u_0^* p^* f_*(\mathcal{F}) & \longrightarrow & u_0^* g_* q^*(\mathcal{F}) \\ \sim \downarrow & & \downarrow \\ p_0^* x_0^* f_*(\mathcal{F}) & & (g_0)_* v_0^* q^*(\mathcal{F}) \\ \downarrow & & \downarrow \sim \\ p_0^*(f_0)_* y_0^*(\mathcal{F}) & \longrightarrow & (g_0)_* q_0^* y_0^*(\mathcal{F}) \end{array}$$

in $\mathrm{QCoh}(U_0)$. To establish the desired claim, it is enough to show that the upper horizontal map is an equivalence, since u_0^* is conservative. The lower horizontal map and upper right vertical map are equivalences by the induction hypothesis, reducing us to checking that the lower left vertical map is an equivalence. This follows from Proposition 1.1.10: setting $Y_\bullet := X_\bullet \times_X Y$ and letting $f_\bullet : Y_\bullet \rightarrow X_\bullet$ denote the projection map, we have that $f \simeq \mathrm{colim}_{\Delta_s^{\mathrm{op}}} f_\bullet$ and hence $f^* \simeq \lim_{\Delta_s} f_\bullet^*$, and the induction hypothesis implies that, for each map $[m] \hookrightarrow [m']$ in Δ_s , we have a commutative diagram

$$\begin{array}{ccc} \mathrm{QCoh}(X_m)_{\leq 0} & \xleftarrow{(f_m)_*} & \mathrm{QCoh}(Y_m)_{\leq 0} \\ \downarrow & & \downarrow \\ \mathrm{QCoh}(X_{m'})_{\leq 0} & \xleftarrow{(f_{m'})_*} & \mathrm{QCoh}(Y_{m'})_{\leq 0}. \end{array}$$

We now prove that 2 holds without the restriction on U . Choose a geometrically faithfully flat map $u : U' \rightarrow U$ where $U' \simeq \mathrm{Spec}(C')$ for some commutative ring C' , and form the further pullback

$$\begin{array}{ccccc} V' & \xrightarrow{v} & V & \xrightarrow{q} & Y \\ g' \downarrow & & g \downarrow & & \downarrow f \\ U' & \xrightarrow{u} & U & \xrightarrow{p} & X; \end{array}$$

let $p' := pu : U' \rightarrow X$ and $q' := qv : V' \rightarrow Y$. We have an induced commutative diagram

$$\begin{array}{ccc} u^* p^* f_*(\mathcal{F}) & \longrightarrow & u^* g_* q^*(\mathcal{F}) \\ \sim \downarrow & & \downarrow \\ (p')^* f_*(\mathcal{F}) & \longrightarrow & g'_*(q')^*(\mathcal{F}) \xrightarrow{\sim} g'_* v^* q^*(\mathcal{F}) \end{array}$$

By the special case settled in the previous paragraph, the lower left horizontal map and right vertical map are equivalences. It follows that the upper horizontal map is an equivalence, which proves the claim, since u^* is conservative.

Finally, we prove 1. Choose a cartesian diagram as in 2 where p is geometrically faithfully flat and $U \simeq \mathrm{Spec}(C)$ for a commutative ring C . By Proposition A.2.4.1, q is also geometrically faithfully flat, and so by Proposition A.2.9.3, $p^* : \mathrm{QCoh}(X) \rightarrow \mathrm{QCoh}(U)$ and $q^* : \mathrm{QCoh}(Y) \rightarrow \mathrm{QCoh}(V)$ are t-exact. Since p^* is conservative and preserves filtered colimits, it suffices to show that $p^* f_* : \mathrm{QCoh}(Y)_{\leq 0} \rightarrow \mathrm{QCoh}(U)_{\leq 0}$ preserves filtered colimits. By the case of 2 that has been proved, this identifies with the composition

$$\mathrm{QCoh}(Y)_{\leq 0} \xrightarrow{q^*} \mathrm{QCoh}(V)_{\leq 0} \xrightarrow{g_*} \mathrm{QCoh}(U)_{\leq 0};$$

here q^* preserves colimits, and g_* preserves filtered colimits by the $n=0$ case of 1 established above. $\square$

§A.3. Unipotence. Based on Proposition A.2.12, we single out the following class of quasicoherent sheaves on geometric stacks and prove a base change formula for their global sections; this is ultimately the key result needed in the body of the paper (in our proof of ??).

Definition A.3.1. For X a geometric stack, we let $\mathrm{QCoh}(X)^{\mathrm{uni}}$ denote the smallest full subcategory of $\mathrm{QCoh}(X)$ that contains the unit object $\mathcal{O}_X$, is stable, and is closed under uniformly left bounded filtered colimits (i.e. colimits of filtered diagrams that lie in $\mathrm{QCoh}(X)_{\leq n}$ for some integer n); we say that a quasicoherent sheaf $\mathcal{F}$ on X is *unipotent* if it lies in $\mathrm{QCoh}(X)^{\mathrm{uni}}$.

Remark A.3.2. Let X be a geometric stack and let $\mathcal{F}$ be a quasicoherent sheaf on X . If we find a filtration

$$0 = \mathcal{F}_{-1} \rightarrow \mathcal{F}_0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_2 \rightarrow \cdots$$

in $\mathrm{QCoh}(X)$ that is uniformly left bounded and such that $\mathcal{F} \simeq \mathrm{colim}(\mathcal{F}_*)$ and $\mathrm{cofib}(\mathcal{F}_{k-1} \rightarrow \mathcal{F}_k)$ is unipotent for each $k \geq 0$, then $\mathcal{F}$ is unipotent.

For example, suppose that $\mathcal{F}$ is known to be left bounded, i.e. n -coconnective for some integer n . By Proposition A.2.9.1, the t-structure on $\mathrm{QCoh}(X)$ is right complete, and so $\mathcal{F}$ is the colimit of the uniformly left bounded filtration

$$0 \simeq \tau_{\geq n+1}(\mathcal{F}) \rightarrow \tau_{\geq n}(\mathcal{F}) \rightarrow \tau_{\geq n-1}(\mathcal{F}) \rightarrow \tau_{\geq n-2}(\mathcal{F}) \rightarrow \cdots$$

Therefore, the above observation implies that, to show that $\mathcal{F}$ is unipotent, it suffices to show that $\pi_m(\mathcal{F}) \in \mathrm{QCoh}(X)^\heartsuit$ is unipotent for all $m \leq n$.

Example A.3.3. We consider Definition A.3.1 in the case $X = \mathrm{Spec}(A)$ for A a commutative ring, so that $\mathrm{QCoh}(X) \simeq \mathrm{Mod}_A$. The following classes of A -modules are unipotent:

1. Recall that the ∞ -category $\mathrm{Perf}(A)$ of *perfect* A -modules is the smallest full subcategory of Mod_A that contains A , is stable, and is closed under retracts. It follows that every perfect A -module is unipotent.
2. It follows from 1 and Lazard's theorem that every flat (classical) A -module is unipotent. And consequently, any A -module that is represented by a bounded complex of flat A -modules is unipotent.
3. Suppose that A has finite weak global dimension, so that every classical A -module is represented by a bounded complex of flat A -modules (e.g. A could be a regular noetherian ring, like $\mathbb{Z}$). Then, by 2, every classical A -module is unipotent, and by Remark A.3.2, it follows that any left bounded A -module is unipotent.

Remark A.3.4. Let $f : Y \rightarrow X$ be a geometrically flat map of geometric stacks and let $\mathcal{F}$ be a unipotent quasicoherent sheaf on X . Then, noting that $f^* : \mathrm{QCoh}(X) \rightarrow \mathrm{QCoh}(Y)$ is t-exact by Proposition A.2.9.3, we have that $f^*(\mathcal{F})$ is a unipotent quasicoherent sheaf on Y .

Example A.3.5. Let X be a geometric stack. Suppose that X is *simply connected*, i.e. that the projection map $p : X \rightarrow \mathrm{Spec}(\mathbb{Z})$ induces an equivalence $\tau_{\leq 1}(X) \simeq \mathrm{Spec}(\mathbb{Z})$, where $\tau_{\leq 1}$ denotes the 1-truncation functor on Stk . Suppose also that p is geometrically flat. (These conditions are satisfied, for instance, when $X = \mathrm{B}^n G$ for G a flat affine commutative group scheme and $n \geq 2$, as in Example A.2.7.) Then, by Proposition A.2.9.3, $p^* : \mathrm{Mod}_{\mathbb{Z}} \rightarrow \mathrm{QCoh}(X)$ is t-exact, and hence restricts to a functor $p^* : \mathrm{Mod}_{\mathbb{Z}}^\heartsuit \rightarrow \mathrm{QCoh}(X)^\heartsuit$; and by Remark A.1.6.2, the latter functor is an equivalence. Combining Remark A.3.2, Example A.3.3.3, and Remark A.3.4, we deduce that every left bounded quasicoherent sheaf on X is unipotent.

Proposition A.3.6. Let $p : U \rightarrow X$ be a geometrically flat map between geometric stacks. Then for any $\mathcal{F} \in \mathrm{QCoh}(X)^{\mathrm{uni}}$, the canonical map

$$\Gamma(U; \mathcal{O}_U) \otimes_{\Gamma(X; \mathcal{O}_X)} \Gamma(X; \mathcal{F}) \rightarrow \Gamma(U; p^*(\mathcal{F}))$$

is an equivalence.

Proof. As functors of $\mathcal{F}$, both source and target are exact, and also preserve uniformly left bounded filtered colimits by Proposition A.2.12.1. The claim follows from this, the definition of $\mathrm{QCoh}(X)^{\mathrm{uni}}$, and the fact that the map is an equivalence in the case $\mathcal{F} = \mathcal{O}_X$. $\square$

Corollary A.3.7. *Suppose given a cartesian diagram of geometric stacks*

$$\begin{array}{ccc} V & \xrightarrow{q} & Y \\ g \downarrow & & \downarrow f \\ U & \xrightarrow{p} & X \end{array}$$

such that p is geometrically flat and $f_(\mathcal{O}_Y)$ is unipotent. Then the canonical map*

$$\Gamma(U; \mathcal{O}_U) \otimes_{\Gamma(X; \mathcal{O}_X)} \Gamma(Y; \mathcal{O}_Y) \rightarrow \Gamma(V; \mathcal{O}_V)$$

is an equivalence.

Proof. The map in question identifies with the composition

$$\Gamma(U; \mathcal{O}_U) \otimes_{\Gamma(X; \mathcal{O}_X)} \Gamma(X; f_*(\mathcal{O}_Y)) \rightarrow \Gamma(U; p^* f_*(\mathcal{O}_Y)) \rightarrow \Gamma(U; g_* q^*(\mathcal{O}_Y)).$$

The first map in this sequence is an equivalence by Proposition A.3.6, and the second is an equivalence by Proposition A.2.12.2. $\square$

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